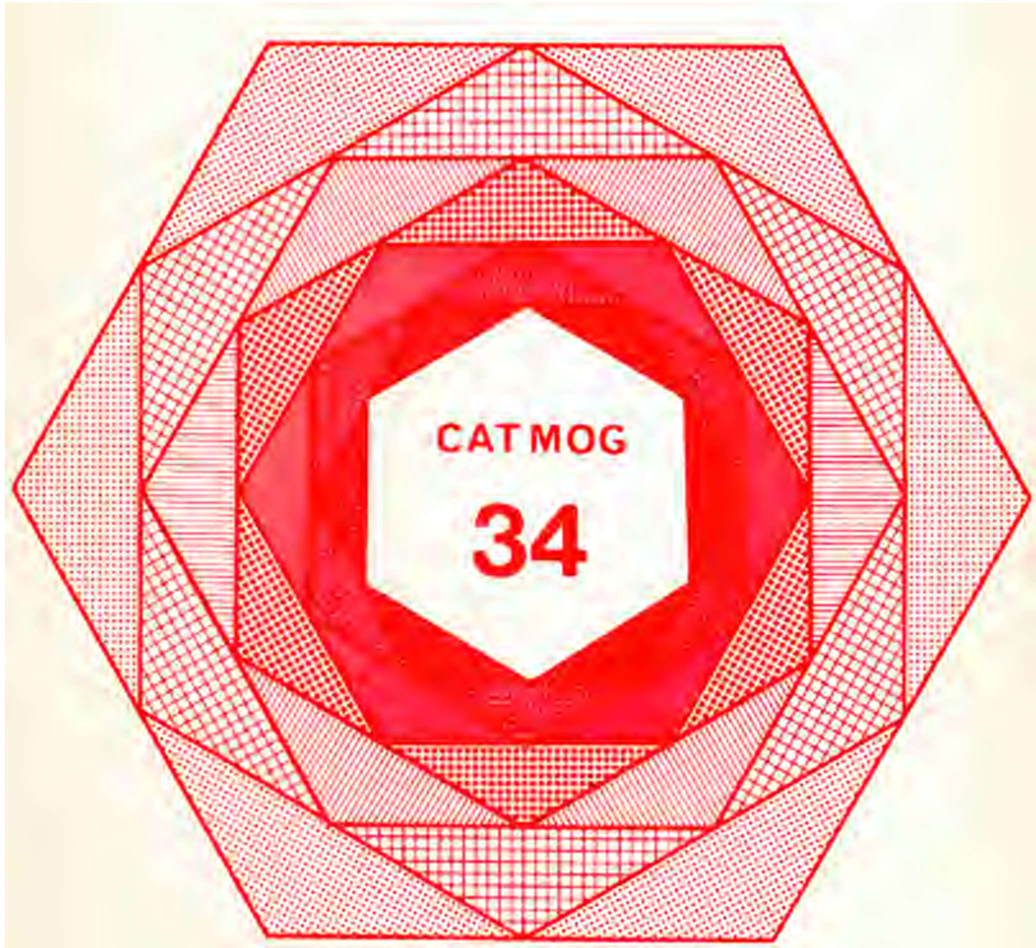
AN INTRODUCTION TO Q-ANALYSIS

J.R. Beaumont & A.C. Gatrell



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AN INTRODUCTION TO Q-ANALYSIS

by

John R. Beaumont and Anthony C. Gatrell

CONTENTS

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Figure 17 is taken from: K.S.O. Beavon (1977) *Central Place Theory: A Reinterpretation* (Longman), with the permission of the author and publisher.

I INTRODUCTION

Much geographical study attempts to describe and interpret the structure of a particular system of interest. Since the essence of structure is the connectivity between individual parts of the system, it is important to be able to describe adequately this basic characteristic. The purpose of this monograph is to provide an elementary introduction to Q-analysis, an operational language of structure devised by the mathematician R.N. Atkin (1974 1977; 1981). It is a language that is characterised by its verbal, graphical and mathematical nature (Gould, 1979), and, at the outset, it is important to appreciate that, unlike the material covered in the majority of CATMOGs, it is not a technique or quantitative method, but rather an entire methodological perspective. In particular, it should be noted that it is completely unrelated to Q-mode factor analysis. It is based on a branch of mathematics called algebraic topology, although it is emphasised that an appreciation of this methodological perspective does not require a deep knowledge of that subject.

Atkin (1974 p. 82) states that the underlying rationale for Q-analysis is '.... understanding the structural properties of relations between sets'. Consequently, the basic ideas of sets and relations are introduced in section II; no prerequisites are required. Given this foundation, the detailed presentation of a worked example, in section III, illustrates the concepts involved in the approach. This example is continued in section IV when specific attention is given to dynamics, attempting to describe and explain structural change. To add force to the demonstration of the potential of Q-analysis in geography, a number of selected applications are considered in detail in section V. In the final section, some of the philosophical underpinnings of Q-analysis are outlined.

As Q-analysis is concerned with describing the inherent structure of data matrices, it is appropriate to briefly recall Berry's (1964) much-cited discussion of a geographic data matrix. To permit a formal ordering of geographic information, Berry suggested the employment of a 3-dimensional matrix (Fig. 1). The rows of this matrix represent geographical locations, the columns represent (physical, social, economic and political) characteristics, and the third dimension represents the same information in a series of time slices. Within this framework Berry showed that ten different types of traditional geographical study can be recognised by considering various operations on the matrix. For example, analysis can focus on a single row (place) or column (characteristic), or, to study areal differentiation or spatial covariance, comparisons can be undertaken of two or more rows (places) or columns (characteristics); in addition, rows and columns can be considered together. These types of analysis can be extended by investigating changes over time. In fact, within this framework, Berry suggests that the contrasts between regional, systematic and historical geography can be interpreted in terms of the three dimensions.

In a later paper, Berry (1968) outlined a general field theory of spatial behaviour which attempted to integrate approaches to formal and functional regionalisation. Particular attention was given to two types of matrix, an 'attribute matrix' and an 'interaction matrix'. Like his geographic data

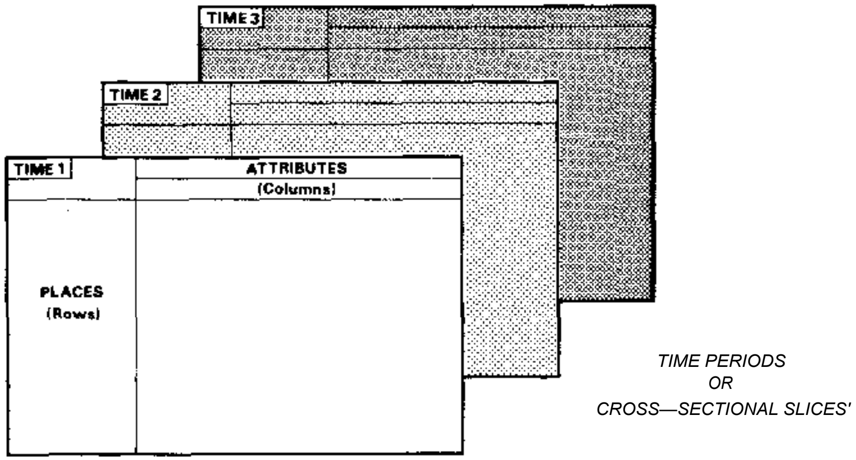


Fig. 1 A Geographical Data Matrix (after Berry 1964)

matrix, the former consists of places (rows) and characteristics (columns). In the latter, if there are n places, clearly, there are $(n^2 - n)$ interactions (or dyads) of interest; these form the rows of the interaction matrix and the columns are types of interaction. Berry suggests that an attribute matrix and an interaction matrix can be factor analysed to produce the underlying dimensions in the form of a 'structure' and 'behaviour' matrix, respectively. In the present context, it is emphasised that the rows and columns are sets. Interest centres on how the rows and columns of each matrix are related. We now proceed to examine more closely what we mean by the notions of a set and relation.

II SETS, RELATIONS AND FUNCTIONS

In this section, a non-technical and informal treatment of some important topics from set theory is undertaken as this is a prerequisite for understanding Q-analysis.

A set is a well-defined collection of elements; it is usually represented by a capital letter, and the elements are often denoted by small letters although in the worked example below we relax the latter convention. Formally,

means that the element a is a member of set A. As Gould (1980a; 1981a) emphasises, the definition of set membership must be unambiguous. It is worth noting the distinction between extensional set definition (where all the elements are listed) and intensional set definition (where all the elements are not listed but defined implicitly). An example of the latter would

be 'cities in the U.K.' and this illustrates how ambiguity may arise. What, for example, is an exact and unambiguous definition of a city, such that we can decide unequivocally whether an urban area is, or is not a member of the set?

In an application of a general set-theoretic description of the structure of central place systems, three particular sets of elements can be deemed important: the set of central places, the set of central place functions, and the set of consumers. Having defined an appropriate study area, and what is meant by the term 'central place', a set of n central places could be written as

$$C = \{c_1, c_2, \ldots\ldots, c_n\}$$

or alternatively as

C {York, Selby, , Wombleton}

Set definition is considered in more detail in conjunction with the worked example described in the next section. It should be noted at this stage that if there are no elements in a particular set it is an 'empty set' (or 'null set') which is denoted by 0.

The idea of a set is extended to a multi-level hierarchical situation through the use of the concept of a cover; that is, different sets can exist at different levels. In general, a set at the N + 1 level acts as a cover of a set at level N, which means that elements of a set at level N may be assigned to more than one set at level N + 1.

It is important to appreciate that this is a more general structure than conventional classifications in which an element at one level can only be a member of one set at the next highest level. The traditional partitions or tree-like diagrams are, therefore, a special limiting case of a cover set; all partitions are covers, but not all covers are partitions (Fig. 2).

Following Atkin (1981), the items ALLOTMENT, GARDEN, NURSERY and PARK can be all thought of as elements of a set and recognisable at the same hierarchical level (say N). Included within these concepts, it is possible to recognise associated phenomena at a lower level (say N - 1) such as FLOWER, FRUIT, LAWN, SHRUB, TREE and VEGETABLE. Clearly, it is possible to list phenomena such as BEAN, BEETROOT, BROCCOLI, ... which would be found in the next level down and so on, to the level of particular species and specimens. Given these two levels, there would likely be overlap between ALLOTMENT, GARDEN, NURSERY and PARK because of the presence of common elements from the N - 1 level. If, however, this was total, they would be synonymous with each other, and, therefore, could be replaced by a single term. It is this overlap which is natural, and it should be explicitly recognised rather than erased by the imposition of a partition.

To give a geographical example of the use of cover sets, Chapman (1981) considers the following set of countries: U.S.A., U.S.S.R., Chile, Mexico, Brazil, Germany, China, France, U.K., and Japan. If the next level up describes continents - America, Europe and Asia (disregarding, for convenience, North and South America and East and West Europe) then there is an apparently obvious relation between the two hierarchical levels (see Fig. 3a). In terms of set membership, however, it is untrue that the whole of the Soviet Union is in both Europe and Asia. It is, therefore, necessary to explicitly name

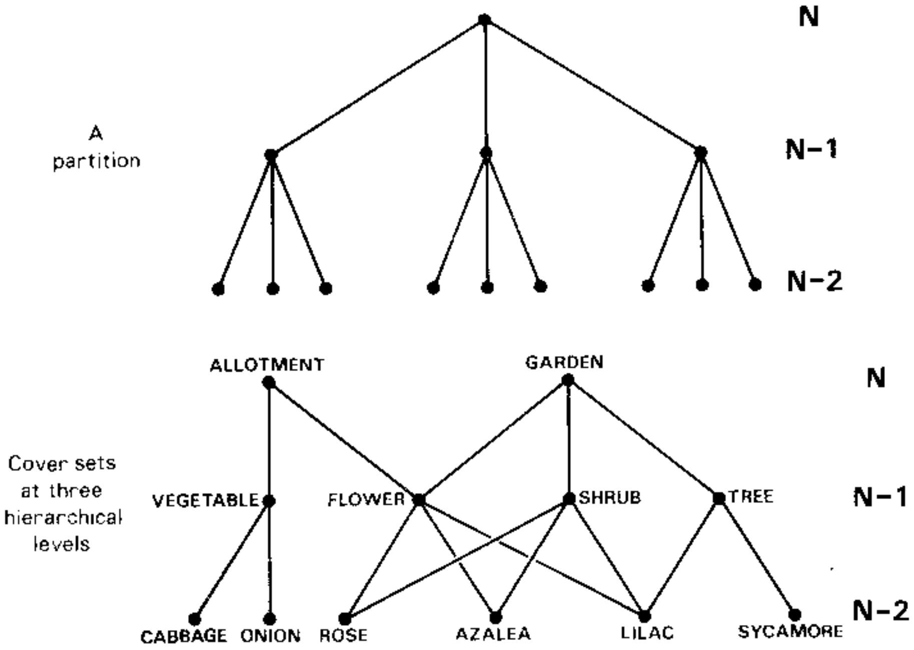


Fig. 2 Partitions and Covers

two parts of the U.S.S.R., European and Asian U.S.S.R. (see Fig. 3b). Consequently, care should be taken when defining sets of interest, and, although such a statement might appear obvious, it is important to note that set theory provides a clear, logical and sound basis for discussion. To repeat: meaningful discussion can only begin once the sets are well-defined.

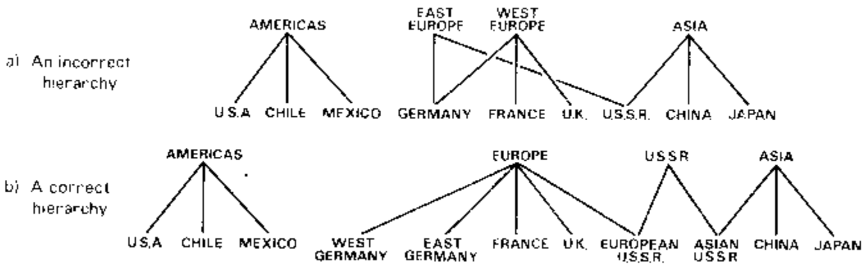


Fig. 3 Geographical Cover Sets (after Chapman, 1981)

The remainder of this section is concerned with one particular level, say N (where N is an arbitrary level). When there are two sets, say A and B, particular interest focuses on the set of all possible ordered pairs of the elements (a, b). This is termed the Cartesian product and represented by

$$A \times B.$$

For example, if

$$A = f1, 2, 3) \text{ and } B = \{1, 2\}$$

then $A \times B = f(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2)\}$.

A relation, λ, is defined as a particular collection of ordered pairs of interest, and, formally, it is written as

$$\lambda \subseteq A \times B$$

indicating that the relation is a subset of the Cartesian product set (the set of all possible pairs of elements). If, for example, set C is defined as a set of centres

C = {LEEDS, LIVERPOOL, LONDON)

and a set P is defined as a set of consumers

P {MRS. BROWN, MRS. COOPER, MRS. JONES, MISS SMITH}

the Cartesian product set is

C x P = {(Leeds, Mrs. Brown), (Liverpool, Mrs. Brown), (London, Mrs. Brown)
(Leeds, Mrs. Cooper), (Liverpool, Mrs. Cooper), (London, Mrs. Cooper)
(Leeds, Mrs. Jones), (Liverpool, Mrs. Jones), (London, Mrs. Jones)
(Leeds, Miss Smith), (Liverpool, Miss Smith), (London, Miss Smith)}.

Since there are three centres we may speak of the i'th centre as c_i (i = 1 2, 3), and we may similarly refer to each consumer as p_j (j = 1, ... 4).

The relation, λ,

$$\lambda \subseteq C \times P$$

'if p_j shops in centre c_i' defines a subset of the Cartesian product set.

For instance:

$\lambda \subseteq C \times P$: {(Liverpool, Mrs. Brown), (London, Mrs. Cooper), (London, Mrs. Jones), (Leeds, Miss Smith)}.

That is, Mrs. Brown shops in Liverpool, and so on.

In Q-analysis, a binary relation is employed to define system structure. That is, the value of λ is either one or zero. For example, in an analysis of the functional structure of a central place system, two sets, a set of centres (C) and a set of goods and services (F), are considered. Since, in general, not all centres will provide all goods and services, not all possible ordered pairs (c_i, f_j) are of interest. A relation, λ, is therefore defined according to whether a particular centre (c_i) provides a specific good or service (f_j); simply stated, c_i is related to f_j if f_j is obtainable from centre c_i. In terms of the binary variable, if c_i and f_j are related in this way the relation is represented by a one, and zero otherwise. A convenient and conventional means to represent λ is using an incidence matrix, Λ (where the rows represent the centres and the columns represent the goods and services). Where the central place system consists of n centres offering up to m goods and services, the associated incidence matrix has n rows and m columns. Formally, it is possible to summarise this information in the following way,

$$A_{ij} = \begin{cases} 1 : (c_i, f_j) \in \lambda \\ 0 : \text{otherwise} \end{cases}$$

A further illustration of a relation between sets is provided in the detailed worked example introduced in the following section.

A brief discussion of the distinction between a function and a relation is now presented. Functions are special and restricted types of relations, and it is noted that most traditional statistical methods are based on (linear) functions. A function is a relation between elements of two sets, usually termed the domain and the co-domain, such that each element of the domain is mapped onto one and only one member of the co-domain. It is this constraint that each element of the domain must have a unique mapping onto the co-domain to be a function that makes such a representation limited in its ability to adequately describe the interconnected nature of geographical phenomena. Given the illegality of bigamy, at least in most countries, it is satisfactory to consider the relation, (' ...... is the husband of ..... ') between a set of married men and married women as a function (see Fig. 4). In contrast, Figure 5 diagramatically reflects a many-to-many mapping between a domain (central places) and a co-domain (retail services); clearly, this is not a function. Thus, analyses based on relations are more general than those using functions:

> ... if we start from the broad definition of the relation, we shall find functions in our data if they are there - because all functions are relations. But if we start from the functional viewpoint, we shall never discover relations - because not all relations are functions' (Gould, 1980a, p.179)

It has, therefore, been argued that conventional multivariate statistical procedures, such as factor analytical techniques (see CATMOGs 7 and 8) suppress a system's inherent structure, because they are founded on functional (usually linear) relationships.

It should be noted that we could relate the elements of a set at one hierarchical level to the elements of a set at a different level. If the hierarchy is a partition then we are dealing, in fact, with a function. More generally, however, if we do not impose a partition on our data, we could analyse the structure of the relations between cover sets. Atkin (1974a; 1978c) gives examples of such analysis.

III Q-ANALYSIS: CONCEPTS AND A WORKED EXAMPLE

(i) Introduction

Consider the map (Fig. 6) of 'Kewtown', which is divided into ten administrative districts. Of these, seven have, located within their boundaries, a single 'public facility'. We know from a growing literature (e.g. Smith, 1977; Cox, 1979; Harvey 1973) that such public facilities may be classified into two broad categories; those, such as recreation centres, hospitals and

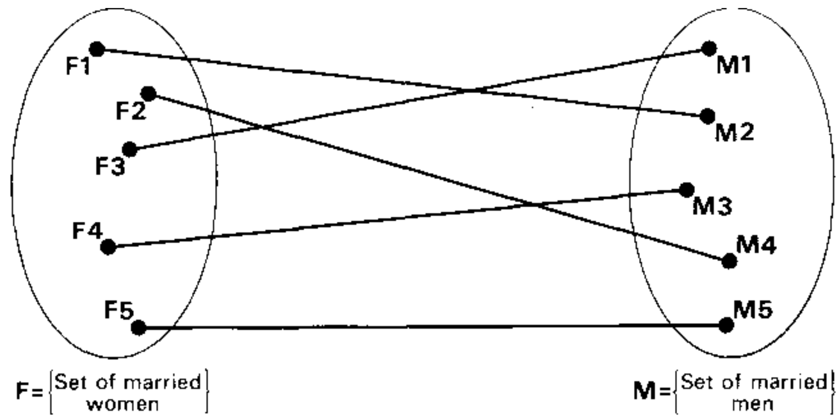


Fig. 4 A function

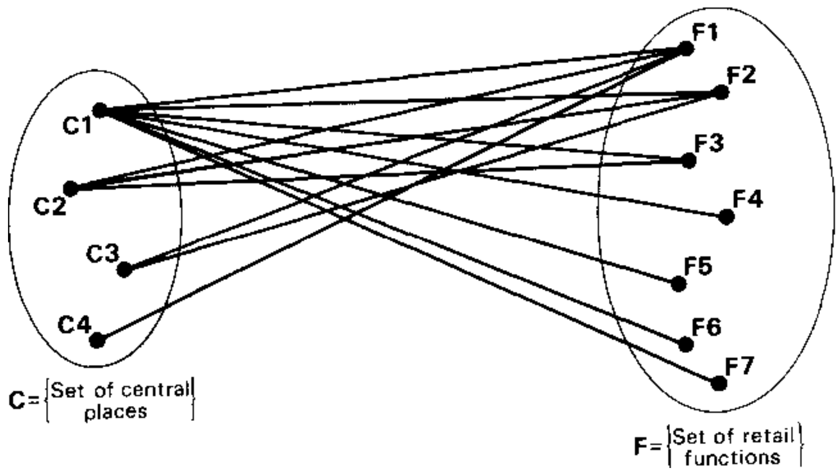


Fig. 5 A relation

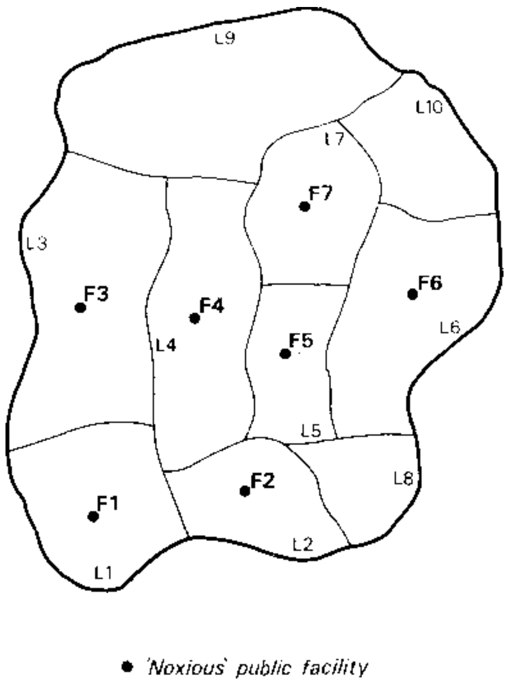


Fig. 6 Kewtown

schools which are generally regarded as desirable by those living relatively close to them (though not, perhaps, by those in the immediate vicinity), and secondly, those 'noxious' facilities (for example, sewage plants, refuse disposal sites and power plants), the presence of which detracts from, or is felt to detract from, the quality of the local environment. We will assume that the seven facilities shown on the map are of the second type.

Formally, we have defined two sets; L, a set of areal units or locations, and a set, F, of noxious public facilities:

L = 1L1, L2, L3, L4, L5, L6, L7, L8, L9, L101

F = {Fl, F2, F3, F4, F5, F6, F71.

No facility exists in areas L8, L9, and L10.

Note that we could define a hierarchy of sets by looking at different spatial scales and by classifying facilities into cover sets, but we restrict attention here to a single hierarchical level.

Now, we can imagine that the facilities 'pollute' their local environment, either by emitting some effluent (smoke, or sulphur dioxide perhaps) or by creating an adverse impact which is less tangible but nonetheless felt by those living in the vicinity. We assume that the pollution is tangible

in what follows, but this is not crucial. We can now be more concrete about notions of 'vicinity' and 'local environment' by defining a relation, λ_A, that indicates that location L_i 'is polluted by' facility F_j:

$$\lambda_A \subseteq L \times F$$

We assume that there is a prevailing wind from the south-west which causes pollution of adjacent zones to the north and east.

We can represent our hypothetical relation in map form (Fig. 7) or, more usefully for analytical purposes, as the incidence matrix Table 1. The relation λ_A tells us that each zone that contains a facility is polluted by that facility (for example, there is a 1 in the cell corresponding to row L2 and column F2) but that the pollutant may carry over into neighbouring zones. We attach to λ a subscript, A, because we shall look below at the same relation but we shall be changing the incidence matrix.

Thus, L2 is polluted by Fl, but F2 does not pollute L1. Further, it is noted that this externality effect, which we build into our relation, is local in that only adjacent zones are polluted; for example, F4 pollutes L7, but L10 is not polluted by F4.

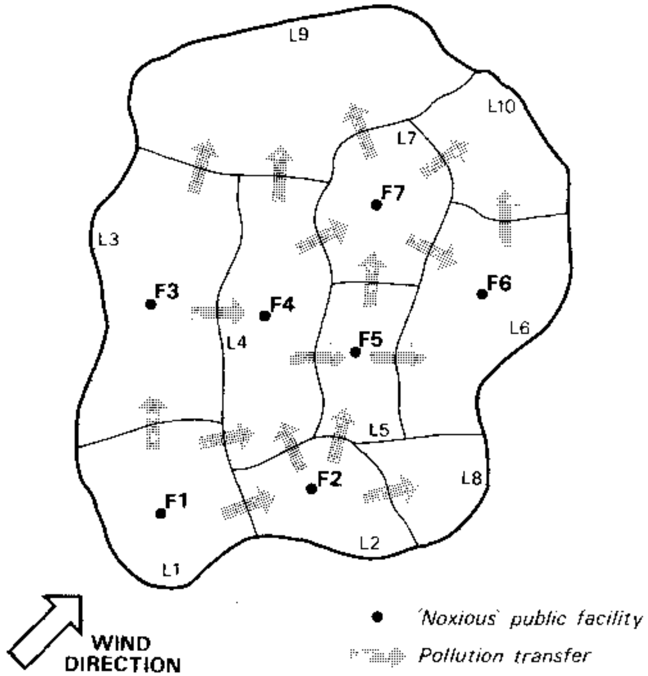


Fig. 7 Spatial Transfer of Atmospheric Pollution in Kewtown

Table 1. Incidence Matrix for $\lambda_A \subseteq L \times F$

λ_A	F1	F2	F3	F4	F5	F6	F7
L1	1	0	0	0	0	0	0
L2	1	1	0	0	0	0	0
L3	1	0	1	0	0	0	0
L4	1	1	1	1	0	0	0
L5	0	1	0	1	1	0	0
L6	0	0	0	0	1	1	1
L7	0	0	0	1	1	0	1
L8	0	1	0	0	0	0	0
L9	0	0	1	1	0	0	1
L10	0	0	0	0	0	1	1

(ii) Simplices and Dimensions

Each areal unit is said to be defined on the set of facilities to which it is related (that is, those which pollute it) and may be represented by a geometrical figure or convex polyhedron called a simplex. The vertices of a simplex are the facilities which pollute that area. For instance, L4 may be depicted as a tetrahedron, defined on the vertices F1, F2, F3, and F4 (Fig. 8a). This may be expressed as:

$$\sigma_3 \text{ (L4)} = \langle F1, F2, F3, F4 \rangle$$

where the angular brackets are used to denote the subset of F that defines the simplex.

We use the Greek lower-case sigma (which should not be confused with its use to represent a standard deviation) to denote the simplex and the name of the simplex is identified in parentheses. The subscript refers to the dimensionality of the simplex. Since a tetrahedron is a three-dimensional figure we say that L4 is a three-dimensional simplex. (It is important to realise that an alternative notation is often employed. Our notation is taken from algebraic topology and follows Johnson (1981a). However, Gould (1980a; 1981a) prefers the notation σ^3_{L4} in which the subscript identifies the simplex and a superscript gives its dimensionality. Those who read the literature should have no difficulty in grasping which style of notation is adopted in a particular piece of work).

We might naturally ask why we could not represent L4 as a square or, for that matter any four-sided figure with vertices F1, F2, F3 and F4 (Fig. 8b). The answer is that there is no reason for joining F1 to F2 and F4 and not also to F3. We could instead draw in diagonal lines to complete the picture (Fig. 8c), but this produces a new 'vertex' which is not a member of F. Another alternative is to draw a two-dimensional figure which joins all pairs of vertices (Fig. 8d), as in a graph (Tinkler 1977), but we want to recognise explicitly that L4 is three-dimensional. Moreover, as we shall see later, the sides or faces of the figure are an important feature of its makeup.

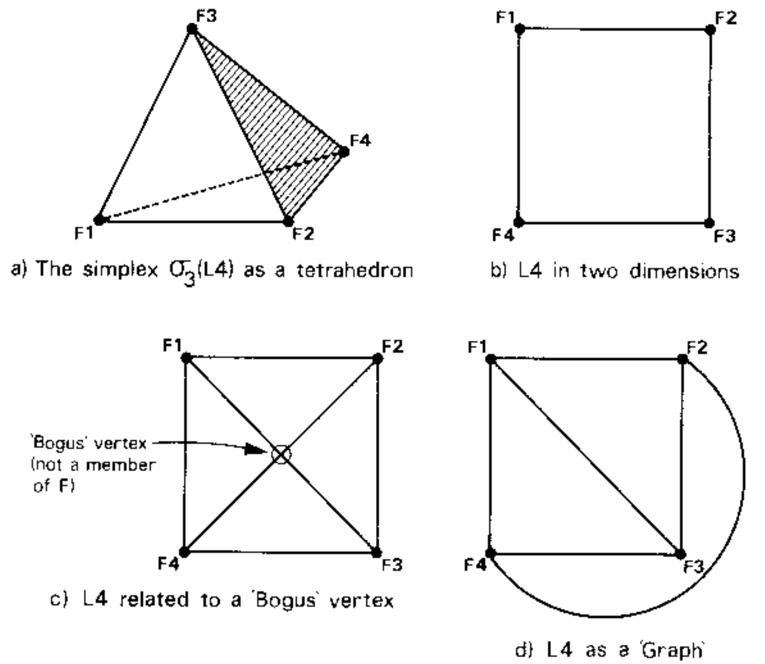


Fig. 8 Alternative representations of an areal unit (L4) polluted by four noxious facilities

Areas L5, L6, L7 and L9 and two-dimensional, represented graphically as triangles, in which the lengths of sides or shapes of the figures are of no consequence. L2, L3 and L10 are defined by two sources of pollution and each is shown as a line. Two zones, L1 and L8, are zero-dimensional, represented by a single point (Fig. 9).

In general, if an area is related to (q + 1) elements of F we say its dimension is q. Given that F contains seven members, q could, in principle, take on a maximum value of six, which would mean that a zone was polluted by all seven facilities. In our example, the maximum dimensionality is three and, although the printed page is two-dimensional, we ought to have little difficulty in 'seeing' L4 as lying or existing in three-dimensional Euclidean space, E^3. However, suppose that L4 was polluted by an additional facility (say, F5), thereby becoming a four-dimensional simplex. Could it be represented graphically? It could (Fig. 10) but the implication is from our diagram that an extra 'vertex' has been introduced (cf Fig. 8c above and Atkin, 1981, pp. 80-83). We need four dimensions, and just because we cannot properly draw four-dimensional figures, we should not be deceived into thinking that such higher-dimensional spaces are any less 'real'.

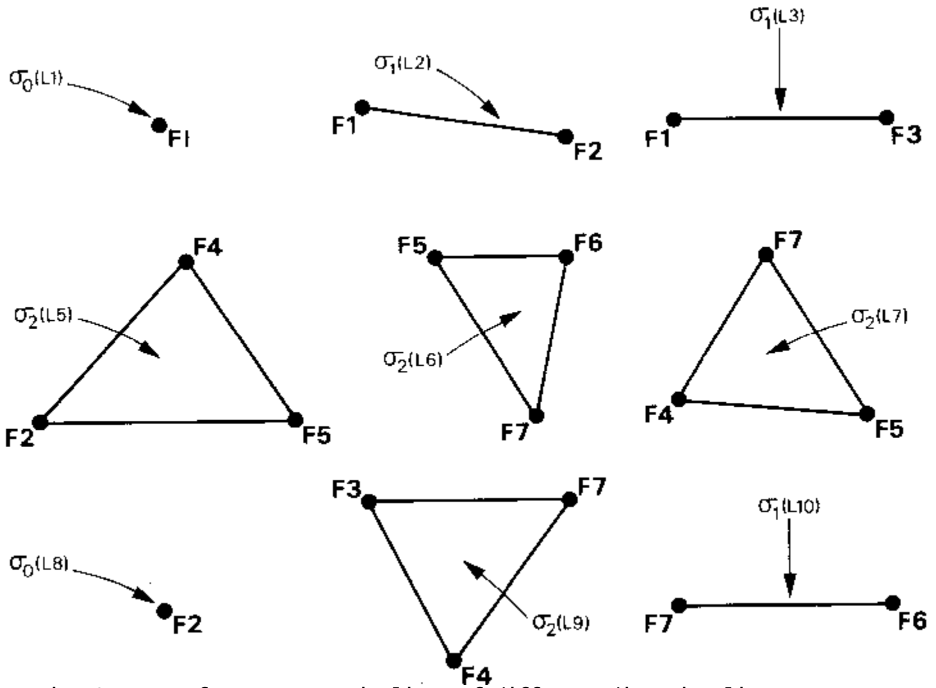


Fig. 9 Areas of Kewtown as simplices of different dimensionality

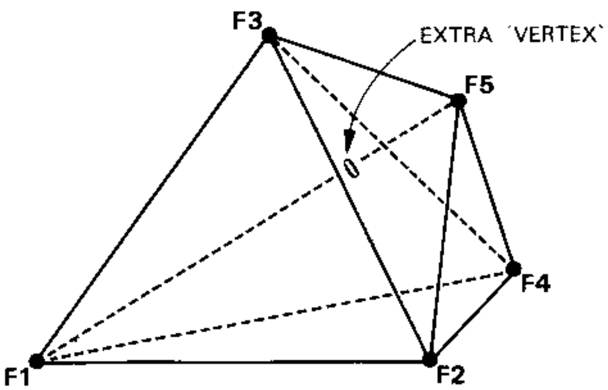


Fig. 10 The impossibility of representing a four-dimensional simplex in two dimensions without introducing a 'bogus' vertex

We may now ask in how many dimensions a simplex lies which is not defined by any vertices? What could we say about a zone which was not polluted by any sources, a row in the incidence matrix which contained all zeros? Such a zone is a <u>null simplex</u> and, by convention, we assign such an area the 'dimension' q = -1.

(iii) <u>Q-Analysis</u>

It is quite evident from Figure 9 that there are simplices which have one or more vertices in common. For instance, L7 and L9 have in common <F4, F7> while L1 has <F1> in common with L2, L3 and L4. This suggests that we can join the ten separate simplices to form a 'complex' of simplices or <u>simplicial complex</u> (Fig. 11). This is written symbolically as K_L $(F;\lambda_A)$, where K denotes the complex with simplices taken from L, the vertices of which are in F, related via λ_A. An alternative notation employed in the literature is K_L $(F;\lambda_A)$, and, in particular papers, there should be no confusion as to which notation has been adopted.

It now becomes clear that the simplices form a connected, multi-dimensional space or structure and it is the connectivities that Q-analysis explores. We now introduce some further terminology.

Since σ_2 (L9) has in common with σ_3 (L4) the vertex set <F3, F4> (a one-dimensional line) we say that it is a <u>1-face</u> of σ_3 (L4). The simplex σ_0 (L1) is defined only on F1 and is a <u>0-face</u> of σ_1 (L2), σ_1 (L3) and σ_3 (L4). We see also that σ_2 (L6) and σ_2 (L7) share vertices F5 and F7 and are therefore 1-faces of each other. In general, if two simplices share q+1 vertices (that is, they share a <u>q-face</u>) we say that they are <u>q-near</u>. Note that, intuitively, we understand by the word 'face' a 'side' of a geometrical figure such as a cube. In algebraic topology, however, it has a more general meaning.

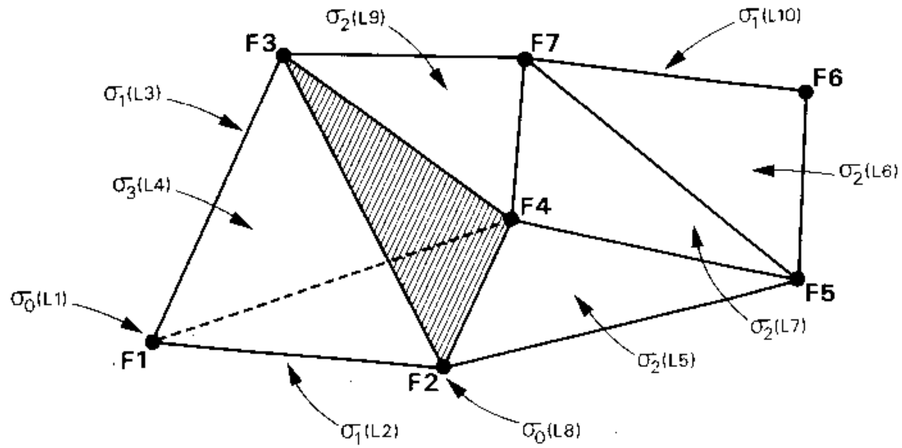


Fig. 11 The simplicial complex $KL(F;\lambda_A)$

We may examine the q-nearness of all pairs of simplices by constructing a shared-face matrix (Table 2). This is clearly a symmetric matrix, the entries of which indicate the 'direct' connections between pairs of simplices. A zero means that a pair of simplices share a single vertex, not that there is no shared face (the latter is represented by -1). The diagonal entries are the dimensionalities of the simplices .

Note, however, that the simplicial complex is in one piece' and that there are therefore 'indirect' connections between all pairs of simplices. These connections or q-connectivities (also called q-chains) exist at different dimensional (0 levels. It is important not to confuse q-connectivity with q-nearness since a pair of simplices may be q-connected but not q-near (on the other hand, q-nearness does imply q-connectivity).

The q-connectivities are revealed formally by a Q-analysis of the complex and this is given in Table 3 (which omits, for clarity, the sigmas).

Table 2. Shared Face Matrix for KL $(F;\lambda_A)$.

λ_A	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
L1	0	0	0	0	-1	-1	-1	-1	-1	-1
L2	0	1	0	1	0	-1	-1	0	-1	-1
L3	0	0	1	1	-1	-1	-1	-1	0	-1
L4	0	1	1	3	1	-1	0	0	1	-1
L5	-1	0	-1	1	2	0	1	0	0	-1
L6	-1	-1	-1	-1	0	2	1	-1	0	1
L7	-1	-1	-1	0	1	1	2	-1	1	0
L8	-1	0	-1	0	0	-1	-1	0	-1	-1
L9	-1	-1	0	1	0	0	1	-1	2	0
L10	-1	-1	-1	-1	-1	1	0	-1	0	1

Table 3. Q-Analysis of KL $(F;\lambda_A)$

q = 3 : {L4}

q = 2 : {L4} {L5} {L6} {L7} {L9}

q = 1 : {L2, L3, L4, L5, L6, L7, L9, L10}

q = 0 : {L1, L2, L3, L4, L5, L6, L7, L8, L9, L10} = {all}

Note: Each component is enclosed by curly brackets

Gould (1981a) has suggested that we can usefully imagine putting on spectacles that allow us to see simplices at a given dimensional (q) level, but not at lower levels, and then putting on (perhaps weaker - or stronger) spectacles that allow us to see simplices at level q and q-1, but no lower. For instance, our three-dimensional spectacles permit us to see only the simplex σ_3(L4). At q = 2 we see five simplices (σ_3(L4) plus four 'new' ones) but they are all disconnected at that level. For example, σ_2(L5) is a 1-face of σ_3(L4) but there are no 2-faces and therefore no 2-connectivities. Since the five simplices at q = 2 are disconnected from each other each forms a separate component.

However, at q = 1 there is a single component which comprises all simplices except σ_0(L1) and σ_0(L8). These eight simplices are therefore 1-connected. Some pairs are 1-near, but σ_1(L2) and σ_1(L10) are 1-connected by σ_3(L4), σ_2(L9), σ_2(L7) and σ_2(L6) (indeed, they are also 1-connected by σ_3(L4), σ_2(L5), σ_2(L7) and σ_2(L6)), although they are not 1-near. At q = 0 the remaining two simplices enter and are connected to all other areal units. Note that as we have come down to the 0-dimensional level we can say that L2 and L10 are 0-connected by σ_0(L8) (which has only just appeared), σ_2(L5), and σ_2(L6). Indeed, there are several 0-chains connecting L2 and L10.

An examination of the structure within a component, such as that within the sole component at q = 1, may be important and gives a perspective on local structure. We can see, by constructing the shared-face submatrix for a component, which of the simplices that are q-connected are also q-near. We can now begin to see too why Q-analysis is of interest in the study of systems, for both are concerned with structure and connectedness. But what does it actually mean in our example to say that two zones are q-near and q-connected?

Here, q-nearness means simply that pollution sources are shared, as is evident from an examination of Figure 11, yet the interpretation of q-connectivity is a little more subtle. Note that σ_1(L2) and σ_2(L9) are 1-connected, even though L2 and L9 are not contiguous areas. The connectivity arises because σ_1(L2) is defined on a subset of the vertices that define σ_3(L4), while σ_2(L9) is also defined on a subset of the vertices of σ_3(L4). The two subsets do not intersect, however, which is why σ_1(L2) and σ_2(L9) are not q-near (in other words, they do not share a single vertex). But they are 'linked' or connected via σ_3(L4). Indeed, the much-polluted zone L4 here acts out a dominant role in the structure, connecting several lower- dimensional simplices. The significance of such connectivities is highlighted subsequently.

If our interest lies more in the structure of the entire complex rather than that of individual components we may speak of global structure and describe this by counting the number of components at each q-level. We can present this information as a structure vector, Q (sometimes called the first structure vector) which is, in our example:

$$\underline{Q} = \{ \overset{3}{1}\ \overset{2}{5}\ \overset{1}{1}\ \overset{0}{1} \}$$

The small numbers over the entries of Q simply act as a convenient guide to the dimensional levels to which those entries refer.

Frequently, we make use of a so-called obstruction vector, $\hat{Q}$, which is obtained simply by subtracting 1 from each entry in the structure vector, Q. The meaning and usefulness of this becomes clear shortly when the notion of 'traffic' is encountered.

Large entries in Q (or $\hat{Q}$) indicate that the structure is a fragmented or disconnected one. The number Q_2 = 5 means here that there are five separate components at the two-dimensional level; at q = 2 these components (each here comprising a single simplex) are disconnected, though at q = 1 the eight simplices of dimension 1 or higher are fused into a single component (Q_1=1). That all simplices are in one component at q = 0 (Q_0=1) means that the complex is 'in one piece' rather than split into entirely disconnected pieces or subcomplexes.

We have talked so far about global structure and local structure. Can we describe the status of an individual simplex within the entire complex? To do this we need to define three more terms which apply to any simplex: top-q, bottom-q and eccentricity. Top-q, written $\hat{q}$, is simply another name for the dimensionality of a simplex, or the dimensional level at which a simplex first appears in the simplicial complex. Bottom-q, $\check{q}$, is the q level at which a simplex first becomes connected in a component with another simplex, or simplices. Put differently, it is the number of vertices, less one, that the simplex has in common with any other, distinct, simplex. Clearly, $\hat{q} \geq \check{q}$, by definition.

For example, top-q for σ_3(L4) is 3 and for σ_1(L2) it is 1. The bottom-q values are 1 in both cases. If we combine these two values it is possible to obtain a description of how well, or how badly a simplex is embedded within the complex. Intuitively, if a simplex is poorly embedded within the complex we might say it is eccentric, and, to capture this, Atkin (1974a) suggested a simple measure of eccentricity :

$$\text{Ecc}(\sigma) = \frac{\hat{q}-\check{q}}{\check{q}+1}$$

although alternative definitions are possible. The numerator gives a measure of 'absolute' eccentricity or individuality while the denominator standardises this to provide a measure of 'relative' eccentricity. This is important, because a difference, $\hat{q} - \check{q}$, of say 3 is clearly much more significant if a simplex is low in dimensionality rather than high. This point should always be kept in mind when comparing and interpreting eccentricity values in actual empirical research. The minimum value that eccentricity can take is zero, and this is obtained when a simplex or dimension q is q-near (or a face of) a simplex of equal or higher dimensionality. In other words, as soon as it 'appears' at a particular dimensional level it is immediately and totally embedded in the simplicial complex. It has no vertices unique to itself, and therefore is not in the least eccentric or distinctive. In the hypothetical example, the simplices σ_0(L1), σ_1(L2), σ_1(L3), σ_0(L8) and σ_1(L10) have zero eccentricity (Table 4). σ_2(L5), σ_2(L6), σ_2(L7) and σ_2(L9) are all equally eccentric since they are two-dimensional and 1-connected to other simplices.

We can imagine a simplex which is completely disconnected from others, even at q=0, and would agree that such a simplex could not be any more eccentric or isolated; this is true regardless of its top-q. We can suggest,

Table 4. Eccentricities in KL ($F;\lambda_A$)

SIMPLEX	ECCENTRICITY
σ_0(L1)	0.0
σ_1(L2)	0.0
σ_1(L3)	0.0
σ_3(L4)	1.0
σ_2(L5)	0.5
σ_2(L6)	0.5
σ_2(L7)	0.5
σ_0(L8)	0.0
σ_2(L9)	0.5
σ_1(L10)	0.0

somewhat perversely, that it is 'connected' at q = -1, which produces an eccentricity of infinity. This accords with our intuition. In our example the simplicial complex is in one piece at q=0 and the simplex with maximum eccentricity is σ_3(L4), the most 'distinctive' or least well connected in the simplicial complex. It is polluted by four sources ($\hat{q}$ = 3), of which at most two are shared with other areas ($\check{q}$ = 1).

Instead of treating the areal units as simplices and facilities as vertices, we might look at the question of structure from the viewpoint of the facilities and treat them as simplices defined by the areal units they pollute. This means that we are 'inverting' the relation and we now write, for the simplicial complex, KF ($L;\lambda_A^{-1}$). The incidence matrix for the inverse relation, λ_A^{-1}, is obtained simply by transposing Table 1. We now obtain the conjugate complex (Fig. 12) that is, the simplicial complex for this inverse relation.

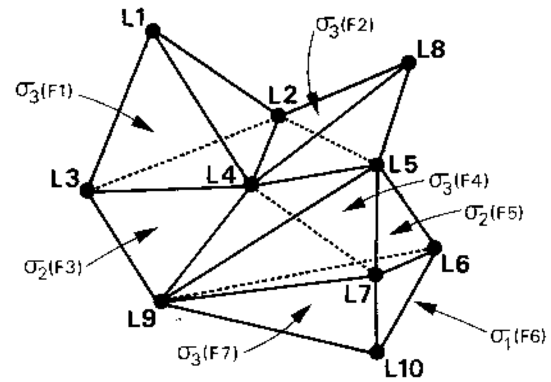


Fig. 12 The Conjugate Complex KF($L;\lambda_A^{-1}$)

There are four 3-dimensional simplices, $\sigma_3(F1)$, $\sigma_3(F2)$, $\sigma_3(F4)$, and $\sigma_3(F7)$ none of which is 3-connected, and, therefore, at q = 3, each forms a separate component (Table 5). At q=2, the simplices $\sigma_2(F3)$ and $\sigma_2(F5)$ enter, but there are no 2-connectivities and so we have six components comprising isolated simplices. At q = 1, all simplices have entered and they are all connected by chains of 1-connectivity.

Table 5. Q-Analysis of $KF(L;\lambda_A^{-1})$

q = 3 :	{F1} {F2} {F4} {F7}
q = 2 :	{F1} {F2} {F3} {F4} {F5} {F7}
q = 1 :	{F1, F2, F3, F4, F5, F6, F7}
q = 0 :	{F1, F2, F3, F4, F5, F6, F7}

$$\underline{Q} = \{\overset{3}{4}\ \ \overset{2}{6}\ \ \overset{1}{1}\ \ \overset{0}{1}\}$$

This tells us that different facilities pollute different subsets of L and thus have distinctive spatial effects. In this simple example, pollution is localised, and, if the 'pollution field' widened, the connectivities at q > 1 would increase. Consequently, more than one facility would be found in components at these levels. Six of the facilities pollute at least three areas, but in no cases are three areas polluted by the same facilities. The eccentricities (Table 6) show that only F6 has zero eccentricity (it is a 1-face of F7), while F1, F2, F4 and F7 are all three dimensional and 1-connected, yielding an eccentricity for each of 1.0. A high eccentricity value indicates that a particular facility has a geographical impact which is quite different from that of other facilities.

(iv) Weighted relations

Q-analysis, we now see, provides us with a means of understanding the structure of a binary relation. Many relations on, or between, sets are however weighted and we need to ask how such relations can be treated, since there may be a combinatorially large number of binary relations that may be extracted. For example, suppose that it was possible to measure the amount of pollution emitted by each facility and the amount transferred from one zone to another over some time interval. In practice, this is likely to be very difficult, but for pedagogic purposes imagine we have such data (Table 7). We further assume that most of the pollution measured within an area (the row total) is derived from within that area, with lower amounts generated externally. The column sums give the total emission levels of facilities and suggest that facilities F4, F5, and F7 are the worst polluters. Denote by x_{ij} the amount of pollution is zone L_i that is generated by facility F_j (e.g. x_{21} = 30, x_{22} = 150, and so on).

Suppose now that the environmental health authorities regard as serious or significant any emission or transfer of pollutant that exceeds some threshold, which we define as θ. If the authorities are strict and regard as significant any pollution or interzonal transfer of pollutant then θ= 1μg/m³ (assuming that this is the lowest observable concentration) and if $x_{ij} \geq \theta$ we allow that L_i is related to F_j. We can imagine that we 'slice' through the matrix $\underline{X}$ at θ =1, and so we call θ a slicing parameter. Consequently we

Table 6. Eccentricities in $KF(L;\lambda_A^{-1})$

SIMPLEX	ECCENTRICITY
$\sigma_3(F1)$	1.0
$\sigma_3(F2)$	1.0
$\sigma_2(F3)$	0.5
$\sigma_3(F4)$	1.0
$\sigma_2(F5)$	0.5
$\sigma_1(F6)$	0
$\sigma_3(F7)$	1.0

Table 7. A weighted relation

	F1	F2	F3	F4	F5	F6	F7
L1	50	0	0	0	0	0	0
L2	30	150	0	0	0	0	0
L3	25	0	140	0	0	0	0
L4	20	50	80	200	0	0	0
L5	0	35	0	75	250	0	0
L6	0	0	0	0	85	180	45
L7	0	0	0	45	45	0	220
L8	0	60	0	0	0	0	0
L9	0	0	35	45	0	0	40
L10	0	0	0	0	0	40	55
Column Totals	125	295	255	365	380	220	360

Note : measurements assumed to be μg/m³

put a 1 in the matrix corresponding to row i and column j, when $x_{ij} \geq \theta$, and a 0 otherwise. This of course yields the binary relation and structure described above (Table 1; Fig. 11).

Suppose instead that the authorities were less strict and monitored as significant only pollution levels exceeding 50 μg/m³. We then set θ = 50 and (using ρ instead of λ to denote that we have sliced a weighted relation) say that $\rho_A \subseteq L \times F$ iff (if and only if) $x_{ij} \geq 50$. This yields a new incidence matrix, (Table 8), a new simplicial complex (Fig. 13), and a new structural description (Table 9).

Table 8. The Matrix $\{x_{ij}\}$ sliced at $\theta \geq 50$

	F1	F2	F3	F4	F5	F6	F7
L1	1	0	0	0	0	0	0
L2	0	1	0	0	0	0	0
L3	0	0	1	0	0	0	0
L4	0	1	1	1	0	0	0
L5	0	0	0	1	1	0	0
L6	0	0	0	0	1	1	0
L7	0	0	0	0	0	0	1
L8	0	1	0	0	0	0	0
L9	0	0	0	0	0	0	0
L10	0	0	0	0	0	0	1

Table 9. A Q-analysis of KL $(F; \rho_A; \theta \geq 50)$

q = 2 : {L4}

q = 1 : {L4} {L5} {L6}

q = 0 : {L1} {L2,L3,L4,L5,L6,L8} {L7,L10}

$\underline{Q} = \{\overset{2}{1}\ \ \overset{1}{3}\ \ \overset{0}{3}\}$

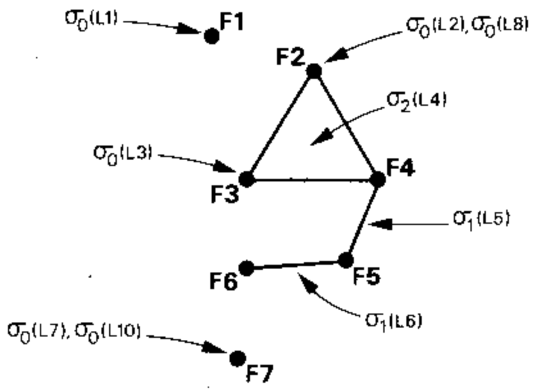


Fig. 13 The simplicial complex $KL(F;\rho_A; \theta \geq 50)$

If we contrast it with the earlier analysis we see that the structure is much more fragmented. (In general, as we increase θ the simplicial complex is reduced in dimensionality and the structure displays less connectedness). Indeed, there are three subcomplexes ($Q_0=3$) and one zone (L9) ceases to be part of the complex since it is not regarded as subject to serious pollution.

We now get a simplex (σ_0(L1)) with eccentricity ∞ (Table 10) indicating its disconnection from all other simplices. Note that σ_0(L7) and σ_0(L10) are also disconnected from other zones, but because they are 0-faces of each other they are not described as eccentric.

As we shall see shortly, such disconnection has fundamental significance for the nature of the protest that may be mounted by the residents of Kewtown.

If the environmental health authorities turn a blind eye to the pollution until it reaches a threshold of 100$\mu g/m^3$ then we can slice the matrix at $\theta \geq 100$. The reader might wish to construct the incidence matrix, sketch the complex that arises and perform the Q-analysis.

Rather than choosing a single slicing parameter which is applied to all zones in Kewtown we might want to give the environmental health authorities some autonomy, allowing each to set its own threshold of 'acceptable' pollution. We then get a set of slicing parameters $\{\theta_i, i=1, \ldots 10\}$ recognising that some authorities may be stricter than others.

In other applications, of course, θ will be column-specific rather than row-specific. Furthermore, we might want to 'slice out' perhaps only a few cells in the matrix, and there are applications when this is appropriate. Since θ controls the system of interest and the structure that emerges, care must be exercised to avoid arbitrary selection of θ. Nonetheless, slicing is a valuable means of creating particular structures and has an important role to play in planning applications. Indeed, it might be of interest to examine how system structure changes as we systematically vary θ.

Table 10. Eccentricities of simplices in KL $(F; \rho_A; \theta \geq 50)$

SIMPLEX	ECCENTRICITY
σ_0(L1)	∞
σ_0(L2)	0
σ_0(L3)	0
σ_2(L4)	2.0
σ_1(L5)	1.0
σ_1(L6)	1.0
σ_0(L7)	0
σ_0(L8)	0
σ_0(L10)	0

We now see that our zones, shown initially as two-dimensional areal units on a map (Figs. 6 and 7) exist in a <u>multidimensional space</u> when related to a set of facilities which pollute them. This multidimensional space may also be described as a <u>backcloth</u>, a stage or setting on which some 'action' or behaviour can take place. Just as the geographer is familiar with the notion of behaviour in two-dimensional space and in particular how behaviour depends upon distances in such space, we now want to try to 'map' various kinds of behaviour onto a multidimensional space or backcloth. We shall see how the connectivities in the backcloth at different dimensional levels present constraints on the behaviour that takes place on the structure.

(v) <u>Traffic and Patterns</u>

"What kinds of 'behaviour' can we imagine existing on the backcloth defined by a relation of zones to pollution-generating facilities? One obvious type of behaviour is the response of the urban residents to the pollution. This response might take the form of protests or complaints made to the environmental health officer in each zone. Atkin calls such behaviour <u>traffic</u> on the structure or backcloth. The backcloth or simplicial complex 'carries' or supports traffic. We stress that 'traffic' is a technical term and that it implies the existence of something on the backcloth. It does not itself imply movement or the 'transport' of anything. (The question of movement is handled by the concept of <u>q-transmission</u> (Johnson, 1980), which we discuss shortly).

Since the simplices in the backcloth are defined in terms of sets of vertices we can imagine that traffic is determined by the vertices. What this means in our worked example is that the level of protest made in each zone depends upon the facilities which pollute that zone, so that we might imagine that relatively high-dimensional zones are capable of supporting more traffic (generating more protests) than low-dimensional, less-polluted zones.

Traffic is represented by a <u>pattern</u>, denoted by Π, which is usually (but not necessarily) a mapping of the <u>simplices</u> in the complex into the set of integer numbers (J). This means that we attach to each simplex a number which denotes the amount of traffic carried by that simplex. For instance, suppose that during a given month we note how many written complaints about pollution are received by each environmental health officer. The complaints received can be regarded as a many-one mapping, Π from $\{\sigma_p\ (L_i)\}$ into J, (Fig. 14), so that L1 generates 2 complaints, L2 generates 5, and so on. Nobody in L8 or L10 registers any complaints.

We return to a consideration of the unweighted relation λ_A. It is useful to write the pattern as a <u>pattern polynomial</u>, which describes the number of complaints associated with <u>the vertices (facilities)</u> that define the simplices (areas). In full, this is written

$$\Pi = 2\langle F1\rangle + 5\langle F1,F2\rangle + 8\langle F1,F3\rangle + 12\langle F1,F2,F3,F4\rangle$$
$$+ 8\langle F2,F4,F5\rangle + 10\langle F5,F6,F7\rangle + 3\langle F4,F5,F7\rangle$$
$$+ 2\langle F3,F4,F7\rangle$$

where $2\langle F1\rangle$ means that two complaints are associated with the simplex $\sigma_1(L1)$ and so on. Moreover, we note that because the simplices have different dimensions then the traffic too is <u>dimensionally graded.</u> To denote the fact that traffic exists at different dimensional levels we may write:

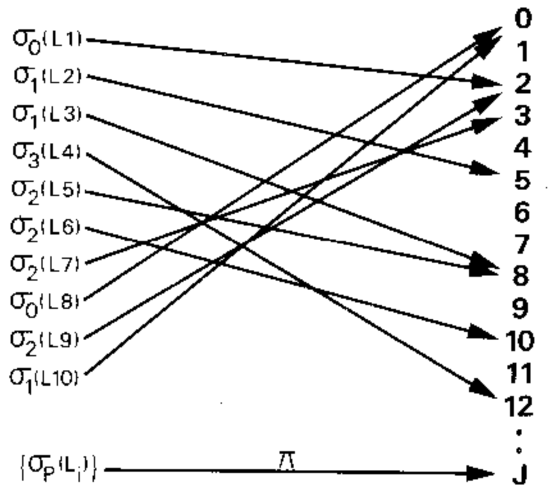


Fig. 14 Pattern as a mapping from simplices into integers

$$\Pi^0 = 2\langle F1\rangle$$
$$\Pi^1 = 5\langle F1,F2\rangle + 8\langle F1,F3\rangle$$
$$\Pi^2 = 8\langle F2,F4,F5\rangle + 10\langle F5,F6,F7\rangle + 3\langle F4,F5,F7\rangle + 2\langle F3,F4,F7\rangle$$
$$\Pi^3 = 12\langle F1,F2,F3,F4\rangle.$$

Thus Π^0 represents 0-dimensional traffic, that which is defined on only one vertex; Π^1 is 1-dimensional traffic, that which is dependent upon two vertices; and so on. Now, the number of complaints registered in any area may not depend on the entire set of facilities that pollute that zone. Traffic may be associated with one or more <u>faces</u> of the simplex, rather than with the entire simplex. For instance, while residents in zone L2 register 5 complaints, they may do so more because they 'perceive' the pollution generated by F2 and remain oblivious to that generated by F1 which spills over into L2.

The 5 complaints thus become 0-dimensional rather than 1-dimensional traffic. To what extent they are <u>only</u> 0-dimensional traffic would be an empirical question.

Further, perhaps the 12 complaints registered in L4 are not all dependent on the entire set of vertices that define $\sigma_3(L4)$ but on different subsets. Perhaps 7 residents are aware of all four pollution sources, 3 'respond to' the two facilities F3 and F4 and 2 are only aware of F4:

$$\Pi\ (\sigma_3(L4)) = 2\langle F4\rangle + 3\langle F3,F4\rangle + 7\langle F1,F2,F3,F4\rangle$$

Again, the precise description we offer depends on the data we collect from respondents.

Q-analysis is able to say something useful about such complaints data because it emphasises the connectedness of the system under investigation. It is the connectivites which may or may not allow traffic to be <u>q-transmitted</u>

through the structure (Johnson, 1980). But as we have seen, these connectivities exist at different dimensional levels (they are q-connectivities) and so traffic of different dimensions (dependent on different numbers of vertices) may be obstructed from moving freely over the backcloth. Let us illustrate this with reference to our hypothetical example, asking firstly what we mean by traffic being transmitted over the backcloth.

Suppose there are people living in L7 who wish to mount a more effective protest against the pollution of their environment. Perhaps on their own they feel somewhat isolated, or perhaps their few complaints are simply acknowledged but filed away by the environmental health officer in L7. They decide that they can be more effective by taking collective action in joining forces with others. With whom can they join? We might hypothesise that their connections to other zone-simplices suggest the formation of an effective pressure group. For instance σ_2(L9) is 1-connected to σ_2(L7), via <F4,F7>, so that we can imagine that as both are polluted by F4 and F7 they decide to tackle the pollution problem together rather than in isolation. This is traffic moving over the backcloth, in Atkin's terminology.

In fact, with the exception of L1 and L8, which are only 0-dimensional, all simplices are 1-connected, which means that 1-dimensional protest traffic, that which requires two vertices for support, can move freely (without obstruction) over the backcloth. We could, in fact, ascertain this from the obstruction vector, where we observe $\hat{Q}_1 = 0$; there is no obstruction at q=1, because all the simplices are in one component. A coordinated protest from eight zones is thus possible, one directed at the regional environmental health officer perhaps. We note that this may be possible even though pairs of zones are polluted by different sets of facilities. For instance, while σ_1(L2) and σ_2(L9) are not q-near (for any q) they are connected via the relatively high-dimensional simplex σ_3(L4).

If area-simplices require only a single pollution-vertex in common with (at least) one other zone in order to act together then all zones are in a position to join forces. 0-dimensional traffic can move freely; it is not obstructed ($\hat{Q}_0 = 0$). Again, the question of whether zones have only to be 'mildly' polluted (by a single facility) for coordinated protests to arise is an empirical matter.

By way of contrast, consider what the structural implications are for traffic if at least three vertices (facilities) are required to be shared in order for protests to merge. Clearly there are several two-dimensional simplices, but we see at q=2 that these form five separate components ($\hat{Q}_2 = 4$). Traffic which is two-dimensional cannot be transmitted on the backcloth. It can exist, on five simplices, but if areas require three shared vertices (a 2-chain) before they join forces then all they can do is mount separate protests.

IV DYNAMICS

Geographers have, over the past few years, begun to construct models which incorporate an explicit dynamic element. Approaches include time series models and space-time extensions (Bennett, 1979; Bennett, 1981) as well as those derived from the analysis of nonlinear differential or difference

equations (Wilson 1981). A useful collection of essays that demonstrates the range of perspectives and methods is provided by Martin et al., (1978), though the interested reader should also consult Parkes and Thrift (1980).

Many of the existing approaches to understanding system dynamics rely on the specification of functional relationships which, as we noted earlier, may be unduly restrictive. Q-analysis offers a different interpretation of dynamics, which we now proceed to demonstrate, continuing with our urban pollution example.

We begin by distinguishing between two kinds of change. Firstly, the traffic may alter, leading to a new pattern. Secondly, the backcloth itself may change as simplices add or lose vertices. We look first at pattern change before examining more fundamental structural change. Consider Table 11, which shows the amount of traffic that exists on each simplex in month T, written as Π. Perhaps, as a result of the q-connectivities among simplices, awareness of the pollution problem grows among residents and, in most of the areas, the number of complaints registered in month T+1 (denoted by Π_{T+1}) has grown (Table 11).

Table 11.

Simplex	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
Π_T	2	5	8	12	8	10	3	0	2	0
Π_{T+1}	2	8	13	17	12	10	5	2	1	1
$\delta\Pi$	0	+3	+5	+5	+4	0	+2	+2	-1	+1

The pattern changes ($\delta\Pi$) may be interpreted as forces acting on the backcloth; where the pattern change is positive, new complaints are attracted to the area and we speak of an attractive t-force. The letter t stands for any non-negative integer (0,1, 2 etc.) and refers to the dimension of the simplex. This emphasises that the pattern changes are associated with particular dimensional levels. For example, we say that the three-dimensional simplex σ_3(L4) experiences an attractive 3-force with a value of +5 (five new complaints), while there is an attractive 0-force of +2 on σ_0(L8). Note that a 0-force is a force felt by a 0-dimensional simplex and does not imply the absence of a force. The t-forces are written in general as $\delta\Pi^t$. Clearly when the number of complaints remains static as in L1 and L6 there are no t-forces, while a repulsive t-force is experienced in L9 where the number of complaints has fallen.

As the level of protest grows in some areas we can imagine that the growth triggers off complaints in other parts of the city. For instance, residents in L4 may start talking to their friends in L3 about the pollution and encouraging them to register protests. Note that L4 is 1-near L3, having F1 and F3 in common as pollution sources.

We can thus imagine the protest spreading or diffusing among zones. But this diffusion will not necessarily be in terms of simple geographical distance as in a Hägerstrand model. We might suggest instead that pattern changes are q-transmitted within a q-connected component (Johnson, 1980). Thus, although

L6, which is not q-near L4, might not be in a position to influence directly the level of protest in L4, the two areas are q-connected via two possible chains of 1-connection (Fig. 11, above) or several possible chains of 0-connection, and we might imagine the 'protest impulse' being transmitted within any of these q-connectivities.

The kinds of changes we have discussed above, alterations in the pattern values, are simply changes in behaviour on a given space or backcloth. Atkin calls this a Newtonian view of change. However, we can also think of the backcloth changing too (an Einsteinian view). This may be illustrated by examining what happens when we close one or more of the public facilities, or introduce new technology to reduce emissions to such low levels that some are no longer regarded as polluters. Such structural changes may well produce changes in previous pattern values.

Suppose, for instance, that the authorities decide to close down F7. How does this affect the structure of environmental pollution in Kewtown? We now get a different subset of L x F, (which we shall call λ_B), embodied in a new incidence matrix, and described by a simplicial complex and Q-analysis (Table 12; Fig. 15; Table 13). Four areas have been reduced in dimensionality. L6 has become 1-dimensional and forms a separate component at q=1. L10 is now 0-dimensional and a 0-face of L6. The simplices σ_1(L7) and σ_1(L9) are still 1-faces (of σ_2(L5) and σ_3(L4) respectively) but they are no longer eccentric because of their reduced dimensionality.

Table 12. Incidence Matrix for $\lambda_B \subseteq$ L x F

λ_B	F1	F2	F3	F4	F5	F6	F7
L1	1	0	0	0	0	0	0
L2	1	1	0	0	0	0	0
L3	1	0	1	0	0	0	0
L4	1	1	1	1	0	0	0
L5	0	1	0	1	1	0	0
L6	0	0	0	0	1	1	0
L7	0	0	0	1	1	0	0
L8	0	1	0	0	0	0	0
L9	0	0	1	1	0	0	0
L10	0	0	0	0	0	1	0

Table 13. Q-analysis of KL(F;λ_B)

q = 3 : {L4}

q = 2 : {L4} {L5}

q = 1 : {L2,L3,L4,L5,L7,L9} {L6}

q = 0 : {all areas}

$\underline{Q} = \{\overset{3}{1}\ \overset{2}{2}\ \overset{1}{2}\ \overset{0}{1}\}$

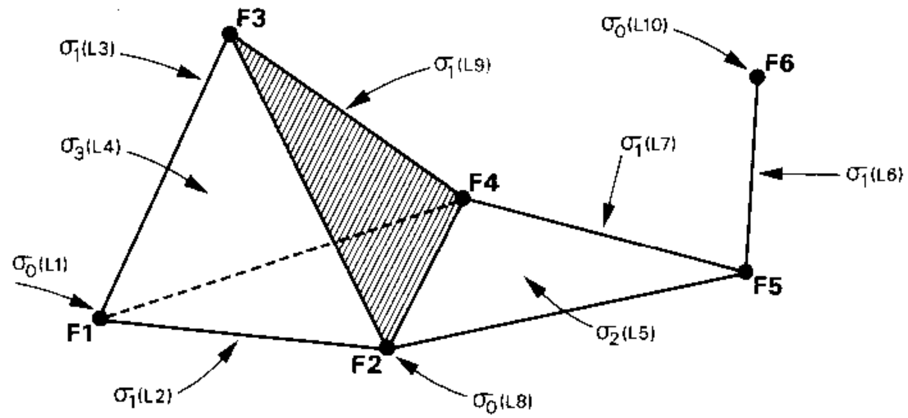


Fig. 15 The simplicial complex KL(F;λ_B)

There are thus now only two 2-dimensional simplices and, in the structure vector, Q_2 = 2 instead of Q_2 = 5 under λ_A. There is less obstruction to 2-traffic, but clearly fewer opportunities for such traffic (protests requiring pollution by three facilities) to exist on the backcloth. However, attention is drawn to the slightly increased obstruction at q=1 (there are now two components), suggesting that 1-traffic cannot move entirely freely on this new structure. The reader might bear in mind the notion that closing down F7 serves to 'fragment' the urban population. It (unintentionally, perhaps) thwarts attempts to form a city-wide pressure group.

Suppose that the authorities decided instead to close down F4 (leaving F7 in place). The incidence matrix (Table 14) yields an interesting geometry which is structurally quite different from λ_A or λ_B, as we might expect given that F4 is a high-dimensional polluter (Fig. 12) in the conjugate complex KF (L;$\lambda_{\bar{A}}^{-1}$). This is clarified by the geometrical representation (Fig. 16) and the Q-analysis (Table 15). There are only two 2-dimensional simplices, the majority existing only at q=1. A calculation of the eccentricities (Table 16) shows that σ_1(L7) is no longer eccentric but that σ_1(L5) and σ_1(L9) are more eccentric than they were under λ_A as they are no longer faces of a higher-dimensional simplex. Broadly speaking, the structure reflects a separation of the city into two 'regions', one focused on L4, the other on L6. The simplices 6_1(L5) and 6_1(L9) act as 1-dimensional bridges connecting these two regions, but the chains of connection are only 0-chains. This creates a feature in the complex which Atkin calls a q-hole and it is a very important feature of a simplicial complex, since it requires that any traffic transmitted on the backcloth has to move around it.

Consider 1-traffic, which can exist on eight simplices in this complex. It can circulate within two components, each containing three simplices and can exist on two other simplices. But it is obstructed from moving freely over the backcloth ($\hat{Q}_1$ = 3), the obstruction being a direct result of the removal of vertex F4 from the backcloth and the consequent creation of a q-hole.

Table 14. Incidence Matrix for $KL(F;\lambda_C)$

λ_C	F1	F2	F3	F4	F5	F6	F7
L1	1	0	0	0	0	0	0
L2	1	1	0	0	0	0	0
L3	1	0	1	0	0	0	0
L4	1	1	1	0	0	0	0
L5	0	1	0	0	1	0	0
L6	0	0	0	0	1	1	1
L7	0	0	0	0	1	0	1
L8	0	1	0	0	0	0	0
L9	0	0	1	0	0	0	1
L10	0	0	0	0	0	1	1

Table 15. Q-analysis of $KL(F;\lambda_C)$

q = 2 : {L4} {L6}

q = 1 : {L2,L3,L4} {L5} {L6,L7,L10} {L9}

q = 0 : {all areas}

$\underline{Q} = \{\overset{2}{2}\ \overset{1}{4}\ \overset{0}{1}\}$

Table 16. Eccentricities of simplices in $KL(F;\lambda_C)$

Simplex	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
Eccentricity	0.0	0.0	0.0	0.5	1.0	0.5	0.0	0.0	1.0	0.0

As both Atkin (1974a) and Gould (1980a) note the q-hole acts as a solid object in the multidimensional space, since traffic cannot pass through it but must circulate around it. Again, this might work to the advantage of the managers of the facilities who wish to see the local pressure groups as fragmented and isolated as possible. We thus see how structural change may be actively promoted in order to serve particular socio-political goals.

Notice that, as the structural backcloth is altered, the traffic is forced to alter in response, in one of two ways. Any 3-dimensional traffic that existed on $\sigma_3(L4)$ in $KL(F;\lambda_A)$ is forced, in $KL(F;\lambda_C)$ to become 2-dimensional or less. The change in the geometry requires that the 3-dimensional protest from residents in L4 becomes 2-dimensional or less, since L4 is now polluted by only three facilities. We say that 3-traffic on $\sigma_3(L4)$

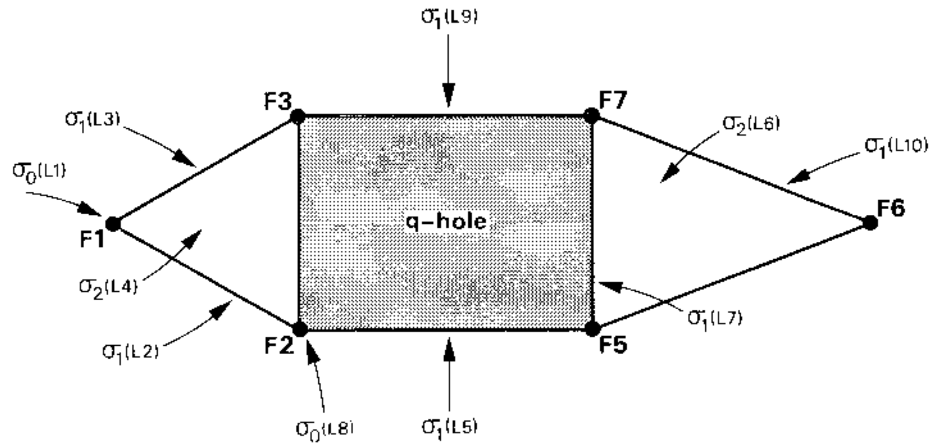


Fig. 16 The simplicial complex $KL(F;\lambda_C)$

experiences t-forces of repulsion. (We saw earlier that t-forces are also produced when pattern changes are recorded). The second way in which traffic may change is in finding other simplices on which it can exist. For instance, in $KL(F;\lambda_A)$ there are five components of separate 2-dimensional simplices, so that 2-dimensional traffic can exist in five areas. By way of contrast, in $KL(F;\lambda_C)$ there are only two 2-dimensional simplices and the opportunities for the existence of 2-dimensional traffic (protests against three sources of pollution) are thus drastically curtailed.

We may, for completeness, mention some other possible forms that this pollution relation can take. One such is realised if each facility is able to 'internalise' the pollution it generates (so that F1 pollutes L1, F2 pollutes only L2, and so on). Three areas become null simplices while the others are disconnected 0-simplices. Traffic of a simple, 0-dimensional kind, may be supported but we can hypothesise that each area mounts a campaign of protest, if any, independently of other areas.

Alternatively, consider two 'extreme' cases; if the facilities are all removed, or the pollution they generate ceases to be emitted, then all areas become null simplices; not a very interesting structure, but presumably the one that the urban residents find most attractive: At the other extreme, all areas within the city might be polluted by emissions from all facilities so that each (with the exception of L8, L9 and L10) becomes a 6-dimensional simplex.

Other possible changes in the urban environment might be to add more than one facility to an area, or (less likely given the capital costs) to relocate a facility from one area to a different part of the city. Depending on precisely how the relation was specified this might or might not change the underlying geometry of pollution.

That we have sketched out a variety of alternative structures from the original sets of areal units and noxious facilities suggests that this perspective has some value as a method for investigating alternative policies for the urban environment. The authorities can produce a series of plans and examine the likely impact on the environment and the extent to which city-wide opposition to plans can perhaps be discouraged. Finally, the example suggests that Q-analysis might be used to study the general structure of urban resource provision, where the set of facilities comprises not noxious ones but rather those which represent public 'goods' and not public 'bads'.

Of course, we have greatly oversimplified a very complex physical and human problem for the purposes of exposition. In reality, while pollution does produce protest, this requires self-confident, educated, concerned and articulate people, who are themselves connected and structured via some social backcloth.

V APPLICATIONS

(i) Introduction

In his recent book, Multidimensional Man, Atkin (1981) propounds the thesis that we all live in a multidimensional space. The subject matter to illustrate Atkin's basic thesis is very wide ranging (being as diverse as chess, poetry, clinical psychology, a Shakespearean play, and so on). In relation to geography, it is interesting to note that much of the development of the theory and practice of Q-analysis was undertaken within a project concerned with the urban and industrial structure of East Anglia (see the various Research Reports entitled, Methodology of Q-analysis, published by the Department of Mathematics, University of Essex, England; Atkin, 1972-1977). More recently, Q-analysis has attracted the attention of a number of geographers, and some of the applications are described in detail in this section. The selection of the specific examples has been to reinforce, in different ways, the ideas that have been introduced in the preceding two sections, and to demonstrate the general potential of this methodological perspective for geography. Although the majority of the studies involve an analysis of the structure of a particular empirically-derived data set, the first discussion is concerned with the familiar theoretical issue about the description of the (hierarchical) structures of Christallerian and Löschian central place systems.

(ii) Service provision

Multivariate statistical techniques have been widely used to classify central places into groups in attempts to derive hierarchies. An alternative approach, using Q-analysis, was proposed by the authors to examine the structure of central place systems in terms of a relation between a set of centres and a set of functions (Beaumont and Gatrell, 1981). It is noted that other structures could have been considered:- market area delimitation can be thought of as the relation of consumers to centres, and the studies of 'baskets' of goods can be thought of as the relation of functions to functions. To summarise this study, some well-known, structural characteristics of the Christallerian and Löschian central place systems, are reconsidered.

The fundamental structural characteristic of the Christaller theory is its rigid successively - inclusive hierarchy; that is, it is assumed that a centre of a specific order n provides functions of order n and below. It is reassuring that the simplicity of this structure is highlighted by Q-analysis, as we now show.

Christaller (1966) suggested three different economic organisational principles, and, although they are usually considered separately, it would be possible to combine them, particularly in an analysis of structural change (see, for example, Parr (1978)). The principles are distinguished on the basis of a 'nesting' factor, K, (not to be confused with the notation for a simplicial complex), which is defined as the number of centres, (including itself) or order n - 1 that a centre of order n serves. The 'marketing' principle (K=3) is the most frequently treated organising principle and it is used in the following illustrations. In a simple, three-order system based on the marketing principle in which there are 9 centres and 6 different functions, one third-order centre (c_1) offers all six functions (f_1, f_2, ...f_6), two second-order centres (c_2,c_3) offer four functions (f_1, f_2, f_3, f_4); and six first-order centres (c_4, c_5, ..., c_9) offer the two lowest-order functions. The incidence matrix for this system (Table 17) provides a complete representation of the relation.

Table 17. Incidence matrix of a Christallerian central place system based on the marketing principle

	f_1	f_2	f_3	f_4	f_5	f_6
c_1	1	1	1	1	1	1
c_2	1	1	1	1	0	0
c_3	1	1	1	1	0	0
c_4	1	1	0	0	0	0
c_5	1	1	0	0	0	0
c_6	1	1	0	0	0	0
c_7	1	1	0	0	0	0
c_8	1	1	0	0	0	0
c_9	1	1	0	0	0	0

Reflecting the ordered structure, the components are:-

at q = 5 $\{c_1\}$
at q = 4 $\{c_1\}$
at q = 3 $\{c_1, c_2, c_3\}$
at q = 2 $\{c_1, c_2, c_3\}$
at q = 1 {all centres}
at q = 0 {all centres}

The associated obstruction vector, $\hat{Q}$, is

$$\hat{Q} = \{\overset{5}{0}\ 0\ 0\ 0\ 0\ \overset{0}{0}\}$$

because the principle of successive inclusion means that a single chain of connection exists. Thus, there is only one component at each q-level. Since it is difficult to represent adequately this 5-dimensional simplicial complex it is hoped that the reader realises that the simplex of the highest-order centre, $\sigma_5(c_1)$, is geometrically the same as the simplicial complex and the geometry of each other centre is a lower dimensional polyhedron or face of the highest-order centre. In general, if a centre provides q + 1 functions, it first appears at the q-level, and centres of the same order offering the same q + 1 functions are q-near. Finally, it is noted that all centres, except the highest-order centre, have zero eccentricity (because 'top-q' equals 'bottom-q', indicating an integrated structure).

As might be expected, a Q-analysis of the Ldschian economic landscape is more complicated. Owing to the symmetry properties of the landscape, it is possible to confine attention to a 60 degree sector of the system (which consists of 325 centres), offering different combinations of the possible 151 functions considered by Lösch (see Beavon, 1977 page 95). Again, attention focuses on the specific functions provided by particular centres; all 325 centres offer at least the lowest-order function (f_0) but only the metropolis (c_1) offers all 151 functions (f_0, f_1, ... f_{150}). Without commenting on the debate concerning the existence or non-existence of a hierarchy in the Löschian system, it is noted that frequently the system is simply represented by only indicating the number of functions offered by each centre (see Fig. 17). This disregards the important characteristic that centres may offer the same number of functions although they are different subsets, and, in so doing, the underlying structure of the Löschian system, especially centre specialisation, is masked.

We chose to disregard the metropolis in our Q-analysis, because as it provides every function a chain of connectivity between all the centres exists. By introducing this modification, the specialisation of the Löschian system is highlighted by the obstruction vector $\hat{Q}$:

$$\hat{Q} = \{\overset{11}{0}, 1, 3, 6, 10, 10, 22, 27, 36, 53, 28, \overset{0}{0}\}.$$

Here, the fact that $\hat{Q}_1$ = 28 means that there are 29 separate components at q=1. Many of these comprise individual centres which offer f_0 plus one other service unique to a centre (e.g. $\{f_0, f_{119}\}$, $\{f_0, f_{131}\}$, and so on). Since all centres offer f_0 there is a single component at q = 0.

The fact that centres offering the same number of goods and services do not necessarily provide identical baskets of goods is illustrated by the apparent fragmentation, with distinct simplices emerging as separate components. Of course, eventually they all fuse into one component at the zero q-level, because each centre provides the lowest-order good. If the concept of traffic is interpreted in terms of consumer trip behaviour, the obstruction vector suggests that opportunities for multi-purpose trips are highly restricted in this theoretical landscape. The spatial distribution of the eccentricity values represents a specialisation surface (Fig. 18) and this appears consistent with the traditional description of 'city-rich' and 'city-poor' areas.

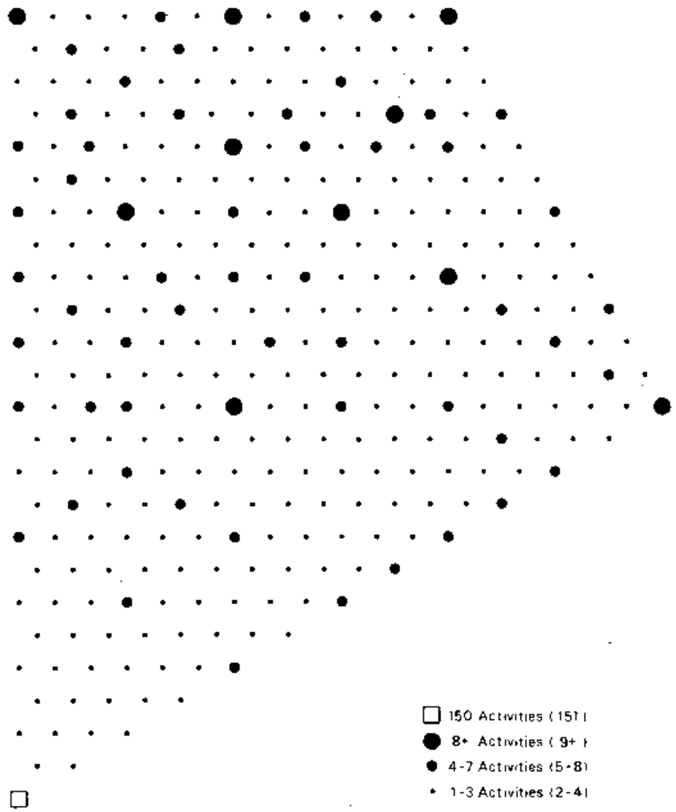


Fig. 17 60 degree sector of the Löschian system (after Beavon, 1977)

The relation, presence or absence of a specific function, is already in a binary form. Often, however, central-place studies involve the collection of information about the number of outlets of a particular function in each centre. Given such a data set, a number of different analyses could be undertaken for different values of the slicing parameters (see Beaumont & Gatrell (1981) for more details).

In terms of dynamics, and following Parr (1978), attention could focus on structural changes in the simplicial complex itself. For example, the K-values for particular orders would alter if the relative 'centrality' of centres altered. In contrast to this Einsteinian viewpoint, a Newtonian perspective would mean that the backcloth remains unaltered and consumers modify their shopping behaviour over time.

Rather than the permanent provision of functions at a given location, periodic markets often exist in peasant societies. On the supply side, simply stated, market periodicity is associated with mobility of individual traders,

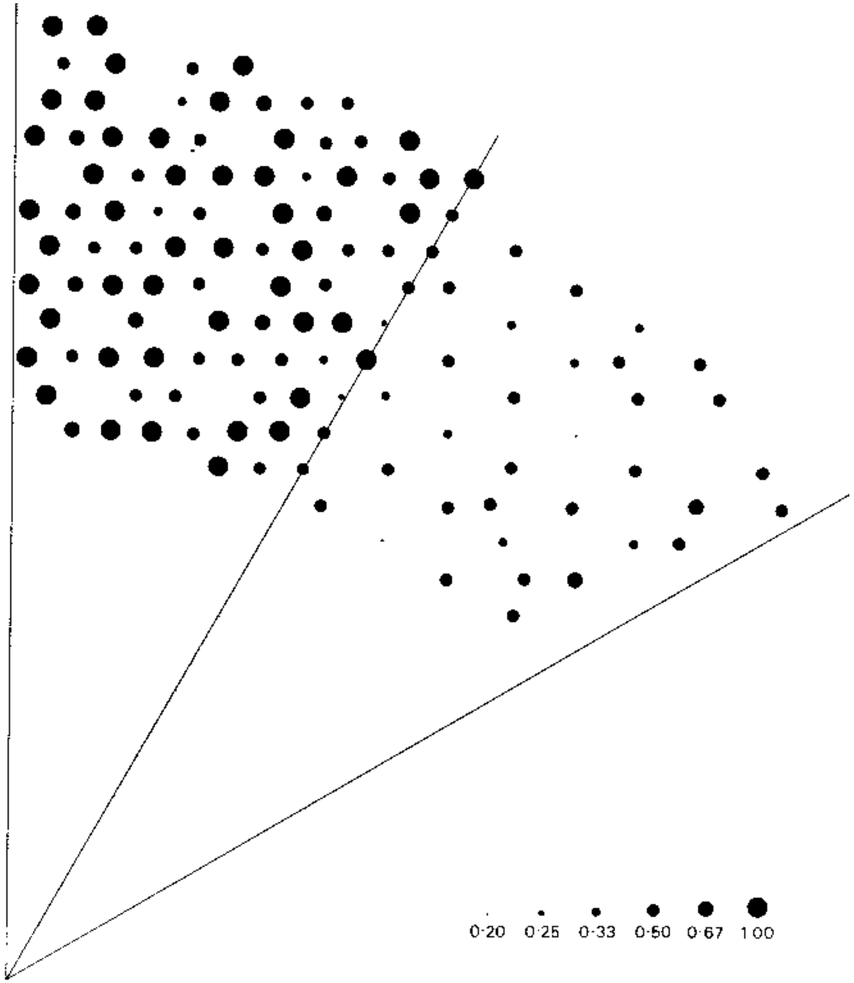


Fig. 18 Eccentricities of central places in the Löschian landscape (from Beaumont and Gatrell, 1981).

and it is frequently considered in terms of space-time synchronisation. Johnson and Wanmali (1981) have undertaken a Q-analysis of a set of periodic markets (M) in India based on their geographical separation, using a slicing parameter representing different cut-off distances. Two markets m_i and m_j were defined as λ-related if the distance between them was less than or equal to θ kilometres. Consequently, the incidence matrix was symmetric. One slicing parameter used was $\theta = 12$ kilometres, because it was assumed that this

is the distance a person would be able to walk in one day. The components at different q-levels were interpreted in terms of the disconnection brought about by physiography and administrative jurisdiction, some of which had deep roots in the colonial past.

Two other models of periodic market structures based on Q-analysis were suggested by Johnson and Wanmali. First, since it is the temporal incidence of occurrence which is the distinguishing structural characteristic of a periodic market, a set, D, was defined as the days of the week. Consequently, the Cartesian product, M x D, of ordered pairs of markets and days, could be defined. The relation was defined as the presence of a market in m_i on a specific day, d_j. In this way, it would be possible to describe the structure of space-time frequency, and consider the suggestion that markets try to avoid competition if possible (see Tinkler (1973)). In terms of dynamics, in particular adopting an Einsteinian perspective, the opening or closing of markets or changing of market day would produce alterations in the backcloth structure. If traffic is interpreted in terms of economic motives, following the Newtonian perspective, consumers' and suppliers' behaviour can be modified through taxation, licencing and so on without altering the backcloth.

A second model considered demand-supply interactions by describing the structure for a distance relation, $\lambda(\theta)$ (similar to one previously discussed) between a set of markets (M) and a set of villages (V), and the interpretation employed common economic forces, such as monopoly of supply and the price mechanism. These forces would affect the economic traffic which can be thought of in terms of the value, quantity and quality of goods sold at a specific market (measured as a pattern). Following the authors' contrived example, it is possible to illustrate q-transmission through the market structure backcloth. Let us suppose that the simplicial complex, $K_M(V;\lambda)$ comprises three 1-connected, simplices (markets); each market is related to four villages, and is therefore, represented by a tetrahedron (Fig. 19). Of particular interest is a simple supply- demand mechanism that illustrates the concept of a q-transmission of t-forces through the backcloth. A price change at market m_1 is transmitted through the 1-connected structure to market m_3 even though these markets share no villages. For example, in response to a relative increase in the price at market m_1, the suppliers from villages v_3 and v_4 may decide to sell a greater proportion of their goods at market m_2. An effect of this increase in the availability of goods at m_2 may involve suppliers from villages v_5 and v_6 deciding to sell a greater proportion of their goods at market m_3. This suggests that geographers might look at the diffusion of price changes in a multidimensional space rather than as a function of geographical distance.

(iii) Social area analysis

In geography, a vast body of literature exists on the structure of urban social areas. Traditionally, factor-analytic methods have been applied to classify areal units into homogeneous social areas, although such conventional partitional procedures fail to capture the complex connectivity of urban social structure.

Indeed,

.... although neighbourhood differences can be observed in most cities so too can overlaps and deviant cases. Thus the study of social areas, both as descriptive mapping and as preparation for predictive analyses, requires

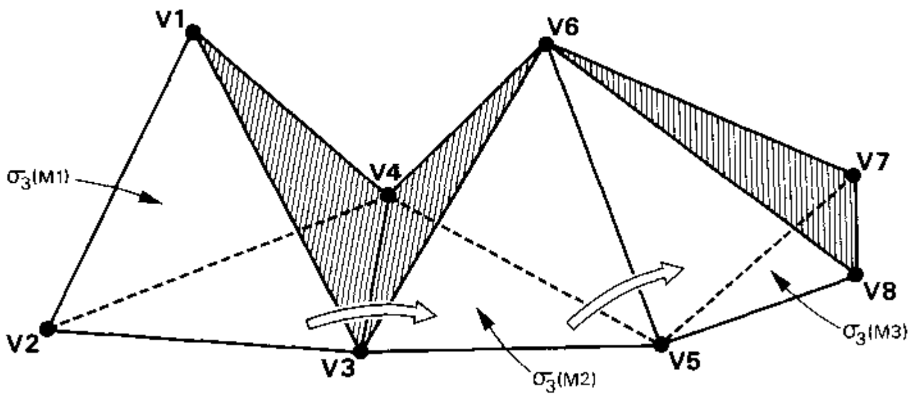


Fig. 19 Q-transmission of price changes through a backcloth of periodic markets (after Johnson and Wanmali, 1981)

a methodology which will unravel the complexities of spatial differentiation between and within districts' (Johnston, 1978, p. 175).

Gatrell (1981a) has explored the possibility of applying Q-analysis, describing the social structure of Thamesdown, Wiltshire, (data for which were published by Johnston (1979)). One set consisted of 25 community areas and the other was 9 occupational groups. The original data set to be analysed gave the percentage of the employed population in each group, and was not, therefore, a binary incidence matrix suitable for Q-analysis. In fact, a family of structures was examined with particular relations defined by the value of the slicing parameter. For example, if θ is defined as

$$\theta \geqslant 20$$

any occupational group containing less than 20 per cent of the employed population in that community area would be ignored (and denoted by a zero). Thus, the incidence matrix replaces percentages of greater than or equal to 20 by is , and the others by Os. As is to be expected, when the value of slicing parameter is decreased, the structure becomes progressively more highly connected. It would also have been also possible to make the slicing parameters occupation-specific, by defining a set $\{\theta_j\}$.

There is a general indication, however, that the community areas have more in common than would be suggested by the application of more conventional taxonomic techniques , and Gatrell (1981a) compares the results of a Q-analysis with those derived by Johnston (1979). Johnston used multi-dimensional scaling (MDS) to map the districts into a Euclidean space of two dimensions, having first constructed a matrix of dissimilarities among all pairs of districts. He also suggested a classification technique for partitioning the set of districts into non-overlapping groups. The several Q-analyses that were performed, on both the original data matrix and on the dissimilarity matrix, cast some doubt on the plausibility of Johnston's taxonomy. However, Gatrell was concerned exclusively with describing what was treated as a

structural backcloth, and no consideration was given to traffic or dynamics.

In an associated study Spooner (1980) suggested that Q-analysis provides an appropriate description of social indicators in Monmouthshire. Given the official desire to identify priority areas of 'multiple deprivation', a variety of indicators, measures of 'need', have been developed and applied in planning. Conventional multivariate statistics have been frequently employed to give a parsimonious description of the structure, but they usually fail to reflect the inherent variety of connectivity between a system's elements and therefore the multidimensional nature of needs. Two sets were defined: a set of 23 Local Authority districts and a set of 17 indexes of 'economic disadvantage', relating to unemployment, rent and rate rebates, receipt of different supplementary benefits, and so on. Again, the new data matrix was in percentage, rather than binary form. Clearly, the selection of the value of the slicing parameter, θ, is not merely a technical issue; indeed as Spooner notes, it is a political decision associated with the general socio-economic values of society and the available finances of the local and central government.

Let us attempt to interpret some of the now familiar aspects of Q-analysis in conjunction with this specific problem. Simply, the q-level at which a particular simplex (or district) appears ('top-q') is an indication of the dimension (or range) of need, and q-nearness reflects the degree of similarity of symptoms present in districts. The eccentricity measures show the extent to which particular districts exhibit distinct combinations of problems. In the conjugate simplicial complex, the indicators are simplices and the districts are their vertices. In this situation, an indicator with an individual spatial distribution has a relatively large eccentricity value, and the structure vector, the summary measure of overall connectivity, gives information on the spatial distribution of each indicator. When there is more than one component at a specific q-level, at least one of the indicators exhibits an individual pattern of incidence. Against these backcloths, traffic and dynamics can be considered in terms of social processes, (particularly social problems) and policy decisions designed to improve conditions.

As Townsend (1979, p. 893) notes, it is important to be able to describe adequately 'the full meaning of the elaborate and interconnected structure of society'. He adds

'Through direct relationships to the economy by virtue of employment and membership of professional and trade unions and through indirect relationships by virtue of membership to income units, households, extended families and neighbourhood, community or regional, ethnic and other social groups, individuals are fitted into a hierarchy of roles'.

This exemplifies Atkin's (1981, p. 13) basic thesis that '.... we all live in multidimensional space'. We leave it to the interested reader to develop a hierarchy of cover sets, consistent with Townsend's remarks. Other studies have been undertaken that compare standard multivariate statistical analyses with Q-analysis. Classifications of cities for example, have been re-examined by Beaumont and Beaumont (1981) and by Chapman (1981).

(iv) Road vehicle management

Johnson (1976; 1981b) has employed Q-analysis in a study of road transport and this methodology is of practical use in vehicle management; indeed, Q-analysis captures the important multidimensional structure of this topic. The particular example is concerned with the problem of congestion at a crossroads, and, following Johnson (1976), two types of simple road intersections are described by Q-analysis.

Remembering the fact that in Britain we drive on the left-hand side of the road, it is possible to define four origins (a, c, e, g) and four destinations (b, d, f, h) (Fig. 20); consequently, it is possible to define a set of twelve origin-destination pairs (or routes)

R {ab, ad, af, cd, cf, ch, ef, eh, eb, gh, gb, gd}

or

$R = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}\}$

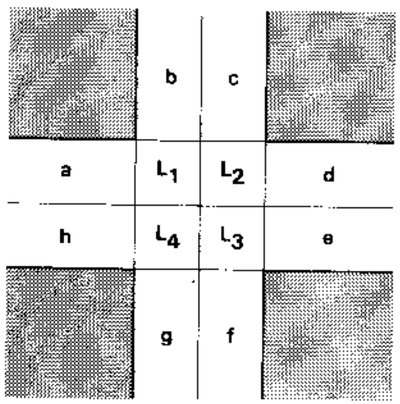
As Figure 20 portrays, another set can be recognised, comprising four 'links' or quadrants within the intersection:

$L = \{l_1, l_2, l_3, l_4\}$

Attention focuses on the relation that describes the links that are used by each route.

The incidence matrix for a narrow junction (Fig. 21; Table 18) is presented in full; a left turning vehicle only traverses one link (l_4) a vehicle travelling straight-ahead traverses two links (l_4, l_1), and a right-turning vehicle traverses all the links (l_1, l_2, l_3, l_4).

The simplicial complex, $K_R(L, \lambda)$, can be graphically represented in diagrammatic form (Fig. 22) with the routes as simplices and the links as vertices. The structure vector, $\underline{Q}$, is a simple unit vector



ROUTE	O-D PAIR
R_1	ab
R_2	ad
R_3	af
R_4	cd
R_5	cf
R_6	ch
R_7	ef
R_8	eh
R_9	eb
R_{10}	gh
R_{11}	gb
R_{12}	gd

Fig. 20 A road intersection in geographical space (after Johnson, 1977)

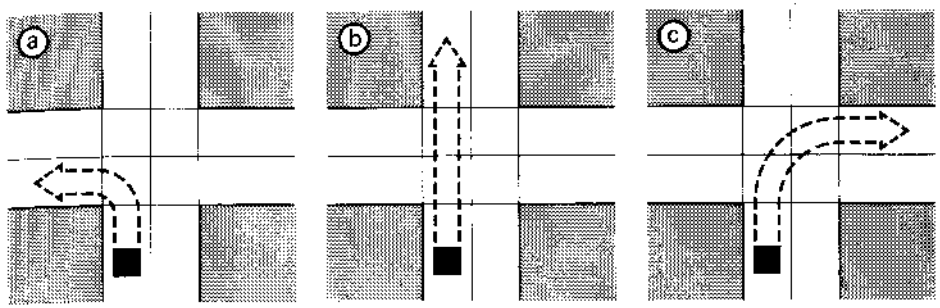


Fig. 21 A narrow junction (after Johnson, 1977)

$$\underline{Q} = \overset{3}{(}1\ 1\ 1\ \overset{0}{1})$$

and the specific components are:

$q = 3$ $\{r_3, r_6, r_9, r_{12}\}$

$q = 2$ $\{r_3, r_6, r_9, r_{12}\}$

$q = 1$ $\{r_2, r_3, r_5, r_6, r_8, r_9, r_{11}, r_{12}\}$

$q = 0$ {all routes}

Table 18. Incidence Matrix for Narrow Junction (after Johnson, 1977)

	l_1	l_2	l_3	l_4
r_1	1	0	0	0
r_2	1	1	0	0
r_3	1	1	1	1
r_4	0	1	0	0
r_5	0	1	1	0
r_6	1	1	1	1
r_7	0	0	1	0
r_8	0	0	1	1
r_9	1	1	1	1
r_{10}	0	0	0	1
r_{11}	1	0	0	1
r_{12}	1	1	1	1

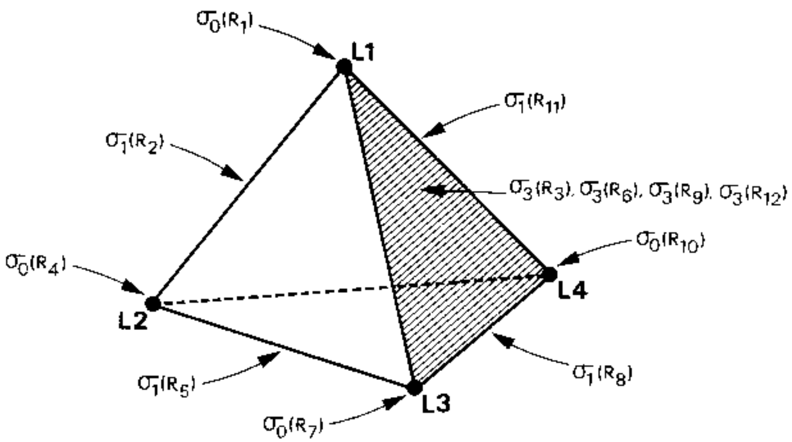


Fig. 22 The simplicial complex $KR(L;\lambda)$

It is also appropriate to describe the conjugate relation, $KL(R,\lambda^{-1})$ in which the links are now simplices. A more disconnected structure is found at high q-levels; the structure vector is

$$\underline{Q} = \{\overset{6}{4}\ 4\ 1\ 1\ 1\ 1\ \overset{0}{1}\}\ .$$

It is important to describe this simplicial complex because it is the backcloth against which patterns of vehicle flow and assignment problems can be examined; in Q-analysis terminology, these management problems are the traffic (which can change over time, because of new routing systems, such as one ways, or alterations in consumer travel behaviour because of real increases in the price of petrol - Einsteinian and Newtonian perspectives respectively). The simplicial complex, $KR(L,\lambda)$, can be thought of as the structure on which consumers can consider the costs (time and money) of different routes, and, therefore, incorporate their perceptions of potential delays because of congestion.

In terms of dynamics, it would be interesting to consider whether congestion itself, is, in fact, a t-force for structural change. Moreover, it can be suggested, at least intuitively, that, in an interconnected road system the removal of a particular link (or links) would have less impact on overall vehicular flow. In terms of traffic management, especially with regard to the practical issue of phasing road maintenance, this aspect would be of fundamental importance.

For comparison a second type of road intersection, a wide junction or mini-roundabout is described (compare Fig. 23 with Fig. 21). The possible routes of vehicles are indicated, and it is left to the interested reader to construct the associated incidence matrix. In contrast to the first type of junction the routes are disconnected at the high q-level. For example, it can be shown that

$$\underline{Q} = \{\overset{2}{4}\ 1\ \overset{1}{1}\}$$

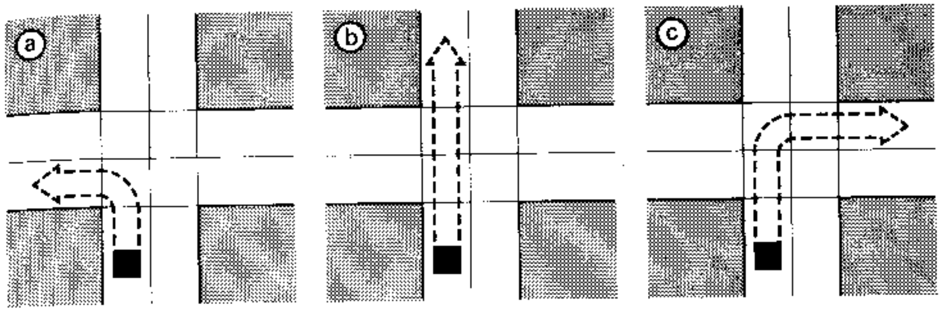


Fig. 23 A wide junction (after Johnson, 1977)

More specifically, the components are

at q = 2 $\{r_3\}$ $\{r_6\}$ $\{r_9\}$ $\{r_{12}\}$

at q = 1 $\{r_2, r_3, r_5, r_6, r_8, r_9, r_{11}, r_{12}\}$

at q = 0 {all routes}

For the conjugate relation, $KL(R,\lambda^{-1})$, the structure vector is

$$\underline{Q} = \{\overset{5}{4}\ 4\ 4\ 1\ 1\ \overset{0}{1}\}$$

and the components are

at q = 5 $\{l_1\}$ $\{l_2\}$ $\{l_3\}$ $\{l_4\}$

at q = 4 $\{l_1\}$ $\{l_2\}$ $\{l_3\}$ $\{l_4\}$

at q = 3 $\{l_1\}$ $\{l_2\}$ $\{l_3\}$ $\{l_4\}$

at q = 2 {all links}

at q = 1 {all links}

at q = 0 {all links}

As a general statement, the wider junction has a structure which is of a lower dimension and is more disconnected at high q-values. Assuming that each link can accommodate a specified magnitude of vehicular flow, it is the more disconnected structure which, in general would suffer less from congestion problems. Moreover, similar problems can be transmitted between links if there is a chain of q-connections; road works in one link, for instance, would probably have impacts on flows on other links if only because of diversions.

Johnson has also described vehicular flows in Colchester. A road network that is highly connected would probably mean that accessibility to different locations would be relatively good, although, as we have seen, if there are any local 'bottlenecks', they are more likely to cause other problems elsewhere. Associated with this idea, it can be suggested that a by-pass route, as the name suggests, enables through traffic to avoid the local town traffic. Johnson found that many of the links in the Colchester by-pass had a non-zero eccentricity value, indicating a degree of integration within the town's road system. In conclusion, Johnson (1977, p.31) stated that

Table 19. N-level elements of the sets, and their N+1 level cover sets (from Gaspar and Gould, 1981)

N LEVEL ELEMENTS	N+1 LEVEL COVER SETS
Backcloth	
Age (years) Literacy (binary) Education (years of schooling)	Personal Characteristics
Parcels Owned (number) Parcels Rented (number) Land Ownded (hectares) Land Rented (hectares)	Land
Small Dam (binary) Well (binary) River (binary)	Water
Small Tractor (binary) Tractor (number) Draught Animals (binary)	Power
Milking Machine (binary) Threshing Machine (binary) Hand Sprayer (binary) Tractor Sprayer (binary) Irrigation Pump (binary) Spray Irrigation (binary) Drop-by-Drop (binary)	Machinery
Cooperative (binary) Wholesale (binary) Factory (binary) Fairs-Markets (binary)	Markets
Neighbors (binary) Technical Advisers (binary) Radio (binary) Television (binary) Others (binary)	Information

'.... the by-pass has become sufficiently connected to the town network, and the links of the by-pass paths sufficiently eccentric for it to have lost its integrity as a by-pass and become part of the town network. If this is so the need for a by-pass to by-pass the by-pass is not surprising.'

A recent, very full summary of this work containing numerous detailed examples, is in Johnson (1981b).

(v) Man-Environment Relations

Man-environment relations are of fundamental interest to geographers, and, recently, Gatrell (1981b) has speculatively outlined the potential of using Q-analysis to describe them. In an important paper, Gaspar and Gould (1981) have employed Q-analysis to describe the structure of agriculture in the Cova da Beira, Portugal where there has been a proposal to construct a large irrigation scheme. A set of 250 farmers, F, and a set of 29 characteristics B, were defined. Table 19 lists the elements of the set of characteristics, which can be thought to exist at level N in an agricultural hierarchy. These elements are also found in N + 1 level cover sets, such as PERSONAL CHARACTERISTICS, LAND, WATER, POWER MACHINERY, MARKETS and INFORMATION (Table 19) but, significantly, no information on CAPITAL at the N + 1 level is available. One possible description of the structure of the region's agriculture is provided by an examination of a binary relation between the set of farmers and the set of characteristics,

$$\lambda \subseteq F \times B$$

where a specific slicing vector, θ_i (i = 1, 2, ..., 29) is applied to each characteristic.

This relation defines an agricultural backcloth on which traffic exists. Interestingly, a set of ten traffic elements, T, was explicitly defined (Table 20), and by comparing two simplicial complexes, $K_F(B;\lambda)$ and $K_F(T,\alpha)$

Table 20. Elements of the Set - Traffic (T)

Apples

Cattle

Cherries

Goats

Maize

Olives

Pears

Peaches

Pigs

Sheep

an indication of the links between backcloth and traffic was derived; crop choice and agricultural practices are governed by social, economic and historical considerations.

Given that the patterns of land ownership greatly influence the structure of agriculture in the Cova da Beira, Gaspar and Gould sliced their data matrix on the basis of age of farmer in order to compare the geometrical structures defined by the youngest quartile of farmers (less than or equal to 45 years old) and by the oldest (greater than or equal to 60 years old). As the dimensionality of these simplicial complexes, $K_B(F,\lambda^{-1})$ differed, to consider their variation in structure it was necessary to use relative q-levels, q_{rel}. Table 21 presents the components for different q_{rel}-levels.

Table 21. Comparison of Youngest and Oldest Farmers in Backcloth Structures (KB(F;λ^{-1}; $\theta \geq 50\%$))

Note : not all q-levels are shown

q_{rel}	Youngest (25%) Farmers	Oldest (25%) Farmers
96	{Irrigation Pump}	
93	{Irrigation Pump},{Literacy, Schooling}	
89	{Irrigation Pump, Literacy, Schooling} {Well}	{Irrigation Pump}
86	{Irrigation Pump, Literacy, Schooling, Well}, {Tractor}	{Irrigation Pump}, {Well}
65	{Irrigation Pump, Literacy, Schooling, Well, Tractor}	{Irrigation Pump, Well}, {Literacy}
64	{Same as above}	{Irrigation Pump, Well}, {Literacy, Schooling}, {Parcels Owned}
61	{Same as above}	{Irrigation Pump, Well, Literacy, Schooling}, {Parcels Owned}, {Hectares Owned}
56	{Same as above}	{Irrigation Pump, Well, Literacy, Schooling, Hectares Owned, Parcels Owned}, {Tractor}
48	{Irrigation Pump, Literacy, Schooling, Well, Tractor}, {Parcels Rented, Hectares Rented}	{Irrigation Pump, Well, Literacy, Schooling, Hectares Owned, Parcels Owned, Tractor}
32	{Irrigation Pump, Literacy, Schooling, Well, Tractor, Parcels Rented, Hectares Rented, Hand Sprayer, Hectares Owned, Parcels Owned}	{All above + Parcels Rented, Hectares Rented}
30	{Same as above}	{All above + Cooperative}
27	{Same as above}	{All above + Tractor Sprayer}
13	{Irrigation Pump, Literacy, Schooling, Well, Tractor, Parcels Rented, Hectares Rented, Hand Sprayer, Hectares Owned, Neighbour, Small Dam, Tractor Sprayer, Other Information, River, Threshing Machine}	{All above + River, Hand Sprayer, Neighbour, Small Dam, Small Tractor, Wholesale Markets}

Note: Underlined simplices enter at that q-level

Source: Gaspar and Gould (1981)

Elements LITERACY and SCHOOLING appear much earlier on the young farmer's backcloth, but the elements PARCELS OWNED and HECTARES OWNED both appear much earlier in the old farmer's backcloth and, therefore, the historical aspects of land ownership may be constraining the young farmers more than the old farmers.

Another distinction can be recognised and interpreted in terms of traffic in the structure of the older farmers, who are primarily involved in commercial fruit farming; the elements TRACTOR SPRAYER and COOPERATIVE appear at relatively higher q-levels, whereas in the structure of the younger farmers, primarily involved with the more extensive farming practices associated with pigs, sheep, cattle, goats and maize, equipment such as a THRESHING MACHINE appears at a relatively high q-level. Although the authors have no information regarding CAPITAL, this does suggest that the young farmers, who are generally better educated, are less constrained by financial considerations than by land ownership. One fundamental corollary is that insecurity undermines the potential of the young farmers who are not really in a position to plan varied and stable activities on a long-term basis. It was suggested that this stress could be interpreted as a t-force changing the structure of agriculture in the region and also the way in which the farmers obtain their livelihood.

Unfortunately, space does not permit us to satisfactorily demonstrate the richness of the above study; the interested reader is strongly urged to read the original paper in full.

(vi) Urban morphology

The phrase 'urban structure' conjures up, for some, the physical fabric of the city rather than the internal residential differentiation which we considered above. Many geographers have been interested in describing the physical form of cities and how such form evolves. This work involves the collection of data on architectural features, building materials, and so on. Atkin has indicated how a more formal analysis of the physical form of towns and street scenes might proceed (Atkin, 1974a). One of his studies looked at the Tudor village of Lavenham in Suffolk, in which the facades of a set of buildings were related to a set of visual features (which included such things as: rendered plaster, solid wooden door, leaded window, wooden lintel, and so on). Rather than consider this example in detail, however, we have chosen to discuss some unpublished research carried out by Atkin and Johnson in the Essex town of Southend-on-Sea.

They considered a single street in Southend and defined a set of 18 land plots, containing properties of various kinds. They further defined a set of 41 architectural features of the properties and defined a relation, λ, between B, the set of plots, and V, the set of visual features. In addition, they had access to a planning proposal which involved demolishing the houses on plots B6-B10 and replacing these by a block of flats. Since the plans showed the visual features of the new construction it was possible to examine another subset, ρ, of B x V. A Q-analysis of both KB(V;λ) and KB(V;ρ), together with their conjugates, gives insights into both the pre-development and the post-development physical structure.

Two features of the Q-analysis are highlighted by Atkin and Johnson. First, the eccentricities of simplices give an indication of how individual plots are affected by the plan (Table 22). We see that in the pre-development structure all but one of the plots have non-zero eccentricity; that is, each has at least one visual or architectural feature which renders it somewhat different from other members of the set, or in some sense 'distinctive'. Replacing the buildings on plots B6-B10 by a block of flats of course ensures that they are defined on a different subset of the features. As a result, the eccentricities of some plots alter; four of them have zero eccentricity, with no architectural features unique to any of those four. However, what is of real interest is that there are several plots in the street which, while remaining standing and physically unaltered, experience a change in their location in the multidimensional space, the street. Each of these plots (83, B11, B12, B13 and B18) register an increase in eccentricity, the result of having redefined the simplices B6-B10. Urban geographers are well-used to the concept of an externality or 'spillover' effect, whereby an object at some location in geographical space has an impact (usually unintended) on a surrounding area (Smith, 1977). In the present example, the changes in eccentricity for undeveloped plots provide a numerical evaluation of the externality effect, an effect which might otherwise be difficult to assess quantitatively.

Table 22. Eccentricities of Plots Before and After Proposed Development (after unpublished work by Atkin and Johnson)

Plot Number	Pre-development	Post-development
B1	0.11	0.11
B2	0.06	0.06
B3	0.16	0.22
B4	0.13	0.13
B5	0.17	0.17
* B6	0.20	0.00
* B7	0.11	0.00
* B8	0.00	0.00
* B9	0.13	0.13
* B10	0.12	0.00
B11	0.05	0.33
B12	0.15	0.35
B13	0.13	0.20
B14	0.11	0.11
B15	0.29	0.29
B16	0.25	0.25
B17	0.13	0.13
B18	0.21	0.42

* denotes buildings facing demolition if the plan is accepted.

The connectedness of the structure of the entire street is described by the structure vectors. These are:

Pre-development

$$Q = \{\overset{22}{1}\ 2\ 4\ 5\ 7\ 7\ \overset{16}{10}\ 8\ 2\ 2\ 2\ \overset{11}{1}\ 1\ \ldots\ldots\ldots\ \overset{0}{1}\}$$

Post-development

$$Q = \{\overset{22}{1}\ 3\ 3\ 6\ 6\ \overset{17}{8}\ 6\ 3\ 2\ 3\ 3\ 2\ 2\ 2\ 3\ \overset{7}{1}\ 1\ \ldots\ldots\ \overset{0}{1}\}$$

The main features to note here are the q-levels at which each complex becomes (and subsequently remains, to q 0) connected into a single component. For the pre-development structure this is q = 11, while it is q = 7 for the post-development geometry. Before the proposed development the structure is a completely connected whole at q = 11; there is a visual 'unity' to the street at this relatively high dimensional level. The plan destroys some of this unity and this only becomes apparent when we move to a lower dimensional level.

The question we may pose is whether this structural change is acceptable, either to the residents or to the planners. We can suggest that Q-analysis provides an appropriate language in which different interest groups can debate planning proposals; once the sets have been agreed and a relation defined, ambiguity is removed and there is a framework for rational discussion. Since much urban planning, particularly the kind of urban preservation issue illustrated here, is concerned with 'soft' systems, it would seem that Q-analysis might well have a useful role to play in aiding decision-making.

VI DISCUSSION AND CONCLUSIONS

We want finally to consider some of the methodological and epistemological features of a Q-analysis, which suggest why Atkin and others find it attractive. We may identify five such features which, although discussed separately for convenience, are clearly interrelated.

(i) An holistic approach

Underlying the concept of a 'system' is the belief that it is the interrelationships or interdependencies among the parts of the system which are of crucial significance in understanding system behaviour and that we cannot comprehend such behaviour by focusing attention solely on the system elements. In a metaphysical sense, this gives rise to the statement that 'the whole is more than the sum of its parts',a statement that epitomises an holistic, Gestalt, or Aristotelian view of phenomena. If a system is regarded as simply the sum of its parts, then some 'additive' analysis will suffice, but if we adopt an holistic stance then 'non-additive' analysis is required (Harvey, 1969, pp 443-5). For instance, ,do we regard a university as a collection of buildings: <library> + <bookshop> + <computer centre> + <Geography Department> + <Sociology Department>, and so on; or do we think of it as an integrated whole which we experience as something more than a collection of individual elements such as buildings?

Q-analysis, with its focus on the structure of a simplicial complex, is holistic rather than atomistic, non-additive rather than additive. Moreover, the notion of structure is implicit in an holistic framework. Q-analysis provides us with a means of representing and describing such structure, and of making well-defined and operational the structure contained in a relation between sets. Given the widespread adoption of a structuralist perspective in numerous areas of human enquiry (Gould, 1975), it comes as no surprise to find Q-analysis being applied to a diverse range of research problems (Gould, 1980a).

(ii) A multidimensional approach

Traditionally, geographers have carried out their investigations in the context of simple Euclidean space. More recently they have begun to recognise the importance of other kinds of space, spaces which are defined and created by relations other than geographical distance (Foyer, 1979). Such relative spaces are frequently mapped in two dimensions, in deference to the geographer's traditional obedience to visual representation. However, such representations may distort the original structure of the relation as, most notably, when two-dimensional configurations from multidimensional scaling are subject to 'stress', or when two factors or components account for only some of the total variance in an original data matrix.

Q-analysis, as we have seen, adopts a very different approach to a relation, retaining the multidimensionality rather than discarding structure in the data. Now, as Atkin (1976, p.493) notes, '.... it is particularly easy to imagine that when we refer to a multidimensional space we are referring to something which in some sense is not 'real". We are conditioned into thinking that three-dimensional space is the space in which we exist and that higher-dimensional spaces are somehow more abstract'. Atkin argues that we do not have to imagine such multidimensional spaces, since they are the spaces we experience and the structure of which Q-analysis seeks to describe.

Underlying Atkin's methodology is, then, the notion that spaces are defined by relations, which contrasts with the Newtonian view of space as a 'container'. Atkin also addresses the notion that time too is more naturally viewed as '.... the manifestation of relations between events' rather than in linear, Newtonian terms as '.... a sort of temporal waste-paper basket, sitting there and waiting for 'events' to exist in it (Atkin, 1981, p.180). Interestingly, this seems to be a direction in which some geographers are moving (Parkes and Thrift, 1980, especially Chapter 2).

(iii) A deterministic approach

Q-analysis is, as we have emphasised, a mathematical language rather than a statistical technique, making no reference to probability theory. It is antithetical to the dominant statistical methodology which characterises much social and natural science. As Gould (1981a, p.297) reminds us, '.... science made excellent deterministic progress for three hundred years before over-enthusiastic proponents implied that statistical methodology was the embodiment of the scientific method itself'. Atkin (1974a; 1981, especially Chapter 6) has discussed the circumstances under which the language of probability theory can meaningfully be applied.

(iv) A 'data-friendly' approach

Given the reliance on the unambiguous definition of sets and the use of relations, Q-analysis keeps us close to the data we collect and to this extent is 'friendly' to the original data. We may contrast this with standard factor-analytic techniques which typically require: data transformation, calculation of correlation coefficients, extraction of factors, and perhaps (orthogonal or oblique) factor rotation and standardisation of factor scores. At each of these stages there are decisions to be made, and this may be followed by numerical taxonomy which further distances us from the data we originally collected. By way of contrast, Q-analysis might be said to 'let the data speak for themselves' (Gould 1981b), and nothing is imposed on the data once the sets and relations have been well-defined.

(v) A 'scientific' approach

We have made little mention here of theories, models or hypotheses and have stressed the primacy of data. Given this, one might be left with the impression that Q-analysis is anti-scientific. This, however, is only true if we view 'science' in a narrow, positivistic sense, in which laws of system behaviour are sought. Q-analysis provides a description of the data whereas we usually think of science providing 'explanations' of phenomena. Yet the distinction between a description and an explanation is not one that can be made simply (Rescher, 1970), and there is a sense in which 'Good description is explanation, for the intellectual content and meaning of the word explanation implies a description of relations between things' (Gould, 1980a, p.171).

This implies that Q-analysis may be an appropriate methodology for 'soft' sciences in which 'hard' quantitative data may be more difficult to collect. Certainly this is the domain of application which Atkin (1981) envisages, where, to repeat, all we require are well-defined sets and relations. Consequently, we should resist attempts to label the method as a 'quantitative technique' and express the hope that, since all areas of human inquiry involve the use of sets and relations, Q-analysis will be welcomed for its fresh perspective.

Some authors (Melville, 1976; Gould, 1980a) go so far as to draw attention to its place in critical social science as an emancipatory language. Whether or not we go along with this view it is clear that Q-analysis permits us to examine for instance the effects on traffic of structural alterations in the backcloth (removal or addition of vertices), which we may then assess. Is it desirable that traffic should be more, or less, obstructed from flowing over a new backcloth? Q-analysis recognises that there is an intimate relationship between 'structure' and 'action'; between 'backcloth' and 'traffic', or between 'space' and 'process' (where space is defined relationally). As a method, then, it seems to offer new insights into a fundamental dichotomy in geography.

Finally, we cannot emphasise too strongly the point that Q-analysis provides a general language which we can use to describe a system of interest. The mathematics from which it comes, algebraic topology, is itself very general. If we follow Olsson (1980) and others in believing that there is an intimate connection between language and the things we can describe and say, then this generality implies that Q-analysis may prove to be a language well worth learning. If, instead, we choose to rely for our descriptions of highly complex, multidimensional human and environmental systems on mathematics more appropriate for descriptions of the physical world, our understanding may prove to be superficial and limited, rather than deep, rich and meaningful.

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