DIRECTIONAL STATISTICS

Gary L. Gaile & James E. Burt

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DIRECTIONAL STATISTICS

by

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I INTRODUCTION

Direction is a geographical primitive. Together with distance it is implicit in all spatial relationships. There are also many instances in geography where explicit observations of direction are of interest. Recent advances in mathematical statistics have provided rigorous techniques which will allow geographers explicitly to incorporate direction in their analyses.

Though arising quite naturally in geographical studies, directional data often behave strangely when subjected to the techniques commonly presented in texts on quantitative geography. Even a technique as basic as computing a mean may lead to disturbing results. Suppose for example two observations of wind direction are taken giving values of 45° and 315° with 0^0 as east. The arithmetic mean of these two observations suggests that the average wind direction is 180° (due west), a result which contradicts the intuitive answer of 0°. At first sight it might seem that the problem lies in the way direction has been measured and that the solution is to transform the numerical system of degrees into one which is suitable for traditional statistical analysis, but in actual fact it is the statistical techniques themselves which require transformation to account for the special properties of angular data. In the example above involving the mean of two angles the results were dependent upon the choice of zero direction. Generally linear statistics can be used with directional data only if the range of observations is small (standard deviation less than 30) and the zero direction problem obviated (Agterberg & Briggs, 1963).

The purpose of this monograph is to present a set of alternative techniques which may be used when traditional analysis is not appropriate. Many of the concepts presented are analogous to those discussed in introductory statistics texts and it is assumed the reader is familiar with those topics usually presented in first-year statistics, together with elementary trigonometry. An excellent sourcebook and the stimulus for what follows is K.V. Mardia (1972).

(i) Fundamental Concepts

Traditional statistics are linear in the sense that variables are considered to be distributed on the number line. Movement from one end of the number line is movement towards the other end. In constructing a histogram, for example, observations are measured with respect to a common (often arbitrary) origin and positions on the line are assigned to each. In directional statistics the observations are angles measured with respect to a reference direction or axis and the unit circle replaces the number line. Each observation is positioned on the circumference of the circle so that a vector joining the center of the circle and the observation forms an angle with reference axis equal to the observed angle (Figure 1).

In collecting observations on angular variables the reference (or zero) direction may be chosen purely for convenience; the only requirement is that all observations be made using the same direction as zero. In this monograph, as in most work in directional statistics, we will take the positive x-axis as zero and measure all angles in a counterclockwise direction from it (see θ_p in Figure 1).

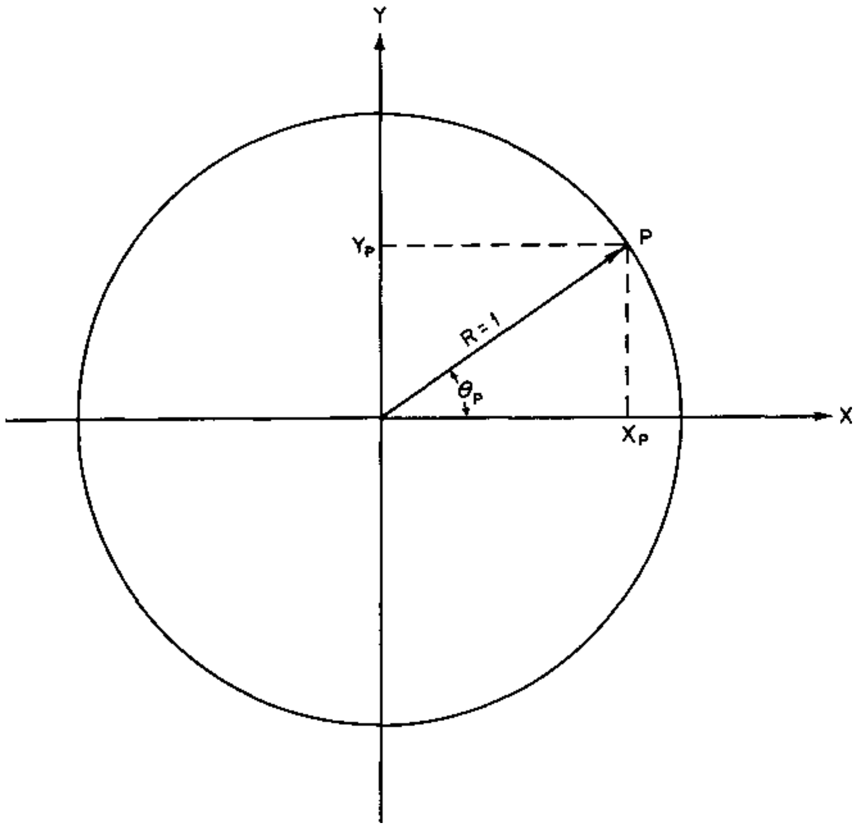

Figure 1. The unit circle and the measurement of an angle.

The unit of measure of angles is also arbitrary so long as it is consistent from observation to observation. In addition to the familiar system of degrees we will express angles in radians. The radian measure of an angle is simply a dimensionless way of expressing its size. If a line segment of length r is rotated about one of its end points through an angle θ, an arc of length S will be swept out. The size of θ in radians is just the ratio S divided by r. If the segment sweeps out an entire circle the length S becomes the circumference of a circle of the radius r or $2\pi r$. There are therefore 2π radians in a complete circle ($2\pi r/r$) and hence one radian is approximately 57.3 (360°/2R).

Consider now an observation P on the circumference of a unit circle - a circle of radius one unit. Its position may be specified by its distance from the centre of the circle (one unit) together with its angular distance from the x-axis (θ). These coordinates, called polar coordinates, are written as the ordered pair $(1,\theta)$. Knowing that θ equal to zero corresponds to the positive x-axis allows one to convert from polar coordinates to the

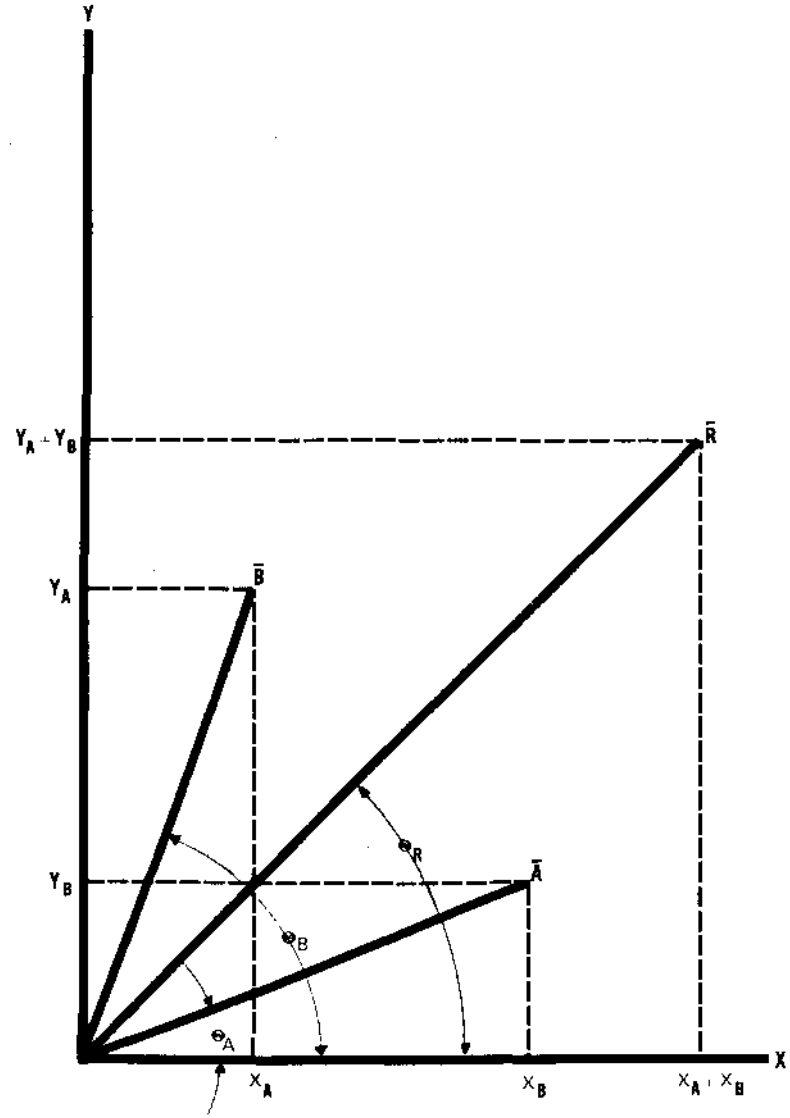

Figure 2. Comparing polar coordinates and Cartesian coordinates.

more familiar Cartesian system. From Figure 2 it is clear that cosine θ and sine θ are the respective X and Y coordinates of P. The observation $(1,\theta)$ in polar coordinates becomes $(\cos\theta, \sin\theta)$ or (X_P, Y_P) in Cartesian coordinates.

The Cartesian coordinates (X_P, Y_P) of P are also the projections of the vector $\vec{OP}$ on the X- and Y-axes. The 'arrow' notation specifies a vector and its direction, in this case from O to P. This notation distinguishes a vector from its length, in this case specified as simply OP. The vector is said to be resolved on the two axes and X_P and Y_P are called the components of vector $\vec{OP}$. In directional statistics observations are treated as vectors of unit length and operations are performed on the components of those vectors. As an example of an elementary operation consider the problem of finding the sum of two unit vectors $\vec{A}$ and $\vec{B}$ having directions θ_A and θ_B.

The mechanics of the situation may be made clearer by considering the vectors to be successive movements of a particle. Vector A corresponds to a displacement of one unit in direction θ_A and likewise $\vec{B}$ a unit displacement in direction θ_B. The sum $\vec{A} + \vec{B}$, called the resultant, can be thought of as giving the position of the particle after the two displacements. It is not difficult to see that the new position will have an X coordinate X_R equal to the sum of the displacements in the X direction and a Y coordinate Y_R equal to the sum of the two Y displacements. In other words the resultant vector R has components:

$$X_R = X_A + X_B = \cos\theta_A + \cos\theta_B,$$

$$Y_R = Y_A + Y_B = \sin\theta_A + \sin\theta_B.$$

Because the resultant vector $\vec{R}$ is also the hypotenuse of a right-angled triangle with the side lengths X_R and Y_R (Figure 2), its length R is given by the Pythagorean Theorem:

$$R = (X_R^2 + Y_R^2)^{\frac{1}{2}} = \left[(\cos\theta_A + \cos\theta_B)^2 + (\sin\theta_A + \sin\theta_B)^2\right]^{\frac{1}{2}} \quad (1)$$

At the same time Figure 2 shows Y_R divided by X_R to be the tangent of the angle between the resultant and the x-axis:

$$\tan\theta_R = \frac{Y_R}{X_R} = \frac{\sin\theta_A + \sin\theta_B}{\cos\theta_A + \cos\theta_B}$$

To find the direction of the resultant, therefore, one need only find an angle whose tangent equals Y_R divided by X_R. Such an angle is called the arctangent of (Y_R/X_R) and is written

$$\theta_R = \arctan(Y_R/X_R) = \arctan\frac{\sin\theta_A + \sin\theta_B}{\cos\theta_A + \cos\theta_B} \quad (2)$$

In determining an arctangent care must be taken to observe the signs of the numerator and denominator as these fix the quadrant of θ_R. An easy way to compute an arctangent is to use a trigonometric table in 'reverse' to find the angle whose tangent equals the absolute value of X_R/Y_R. This will be

an angle between 0 and 90 degrees. A constant is then added to the angle according to the signs of X_R and Y_R as indicated in the table below. For example, the arctangent of (3/-5) is 31° + 90° or 121°.

Table 1. Constants to be used in determining the arctangent of an angle.

Sign of Denominator	Sign of Numerator +	Sign of Numerator -
+	0 Quadrant I	270 Quadrant IV
-	90 Quadrant II	180 Quadrant III

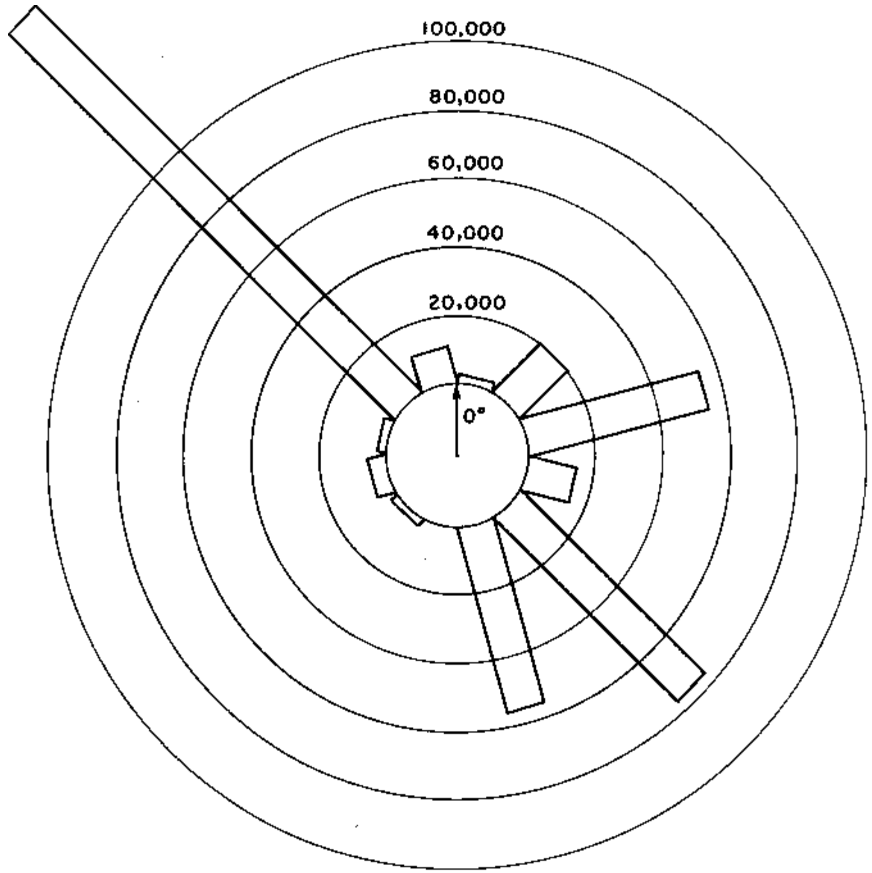

Figure 3. A circular histogram of the Nairobi data.

II DIAGRAMMATICAL REPRESENTATION

Directional data may be graphically represented in various ways. Ungrouped data may be depicted as points plotted on the perimeter of the unit circle or by vectors drawn from the origin to the perimeter. For the more common case of grouped data, circular histograms, linear histograms, and rose diagrams are alternative representations.

Table 2. Frequencies and directions of mail sent from Nairobi during one week for destinations within Kenya.

Direction	Frequency	Direction	Frequency	Direction	Frequency
17.25	814	121.83	1688	162.53	6983
46.31	524	126.47	494	165.38	943
47.60	455	126.63	7652	222.83	1736
53.30	8791	127.12	21373	265.00	1499
54.16	9128	127.87	1737	265.10	2410
54.29	1080	128.99	494	293.71	294
61.31	6071	133.71	1770	298.30	73
62.82	504	135.68	14265	298.50	66
64.75	2188	136.85	5006	300.65	395
65.14	4681	137.59	1790	302.80	123
67.38	5703	138.10	707	305.54	650
73.66	8995	138.85	980	308.16	390
80.31	956	141.12	2940	309.44	3604
80.54	1817	142.94	5383	312.42	50
81.53	8763	146.13	4307	313.17	682
83.50	774	148.19	5391	314.23	647
85.39	13936	150.78	33124	314.58	557
106.14	1149	154.34	6598	315.31	151188
106.70	7008	158.20	5165	329.34	1346
108.97	5312	160.13	926	330.95	8370
113.45	410	162.19	2700	356.99	2311

Table 2 gives the frequencies of mail sent from Nairobi during one week for destinations within Kenya. The data are grouped according to direction of destination with 0° as east. The circular histogram for these data is shown in Figure 3. To construct a *circular histogram*, concentric rings representing frequencies are drawn around a reference unit circle and intervals are chosen in the usual manner. Blocks are extended from the reference circle perimeter with heights equalling observed frequencies and widths equalling the chord subtending the intervals arc on the reference circle.

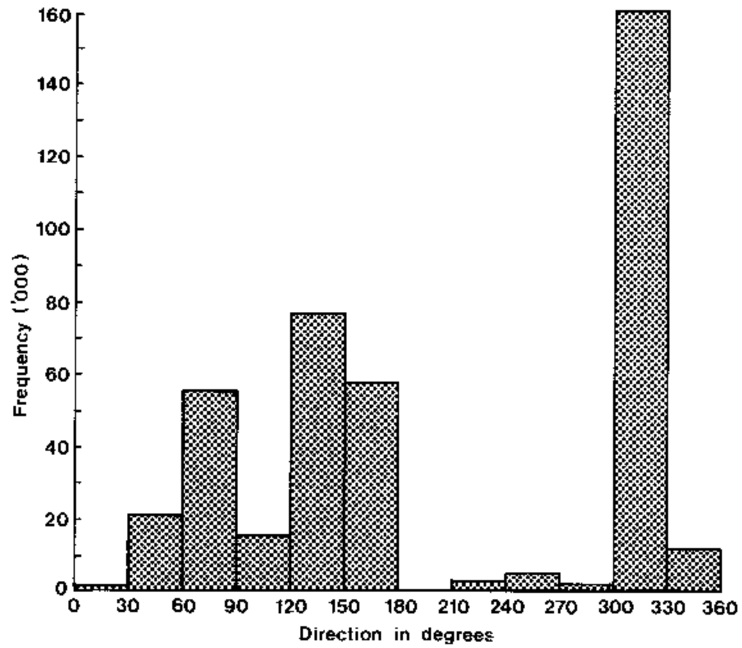

Figure 4. A linear histogram of the Nairobi data.

Linear histograms (e.g. Figure 4) can be considered as unwrapped circular histograms (care must be taken in selection of the cut point, especially in multimodal cases).

Rose diagrams (e.g. Figure 5) are the predominant form of diagrammatical representation of directional data. To construct a rose diagram, radii proportional to the class frequency are extended from the origin along the class interval boundaries and are connected by an arc centered on the origin. The area for each sector thus varies with the square of its frequency. The rose diagram may be modified to make areas of sectors proportional to frequencies by using the square roots of the frequencies to determine radii lengths.

We now wish to summarize the properties of these distributions.

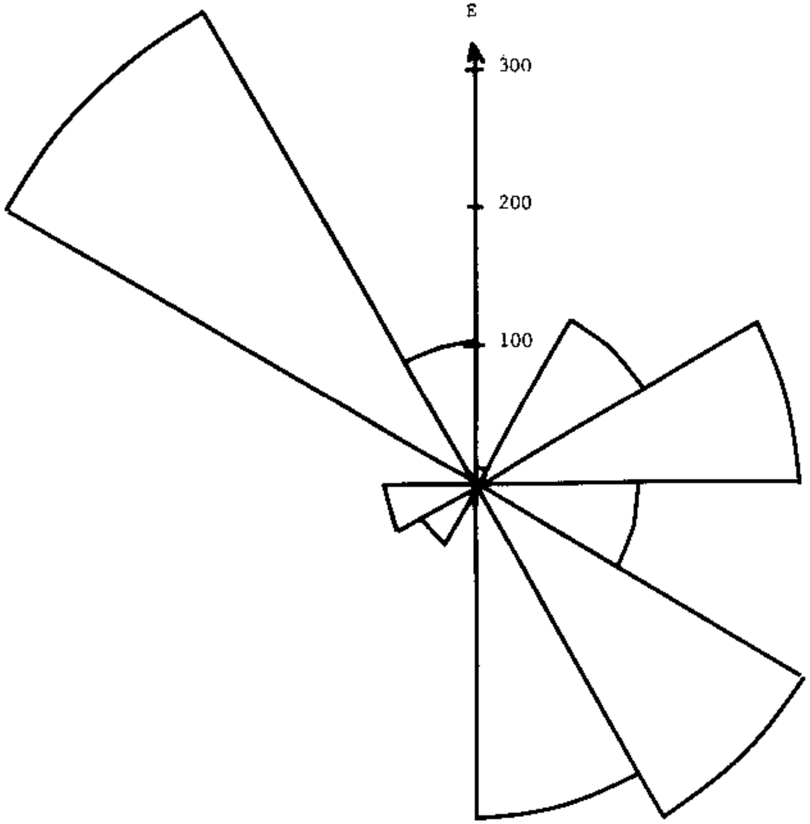

Figure 5. A rose diagram of the Nairobi data.

III DESCRIPTIVE MEASURES

The example involving the mean of two angles showed that statistics calculated using linear techniques are not appropriate in characterizing a sample of angular observations. In this section several measures are presented which can be used as directly analogous to measures which should be familiar from linear statistics.

(i) Directional Mean and Circular Variance

It is ironic that the first arithmetic mean on record was calculated for a set of compass readings by William Borough in 1581. These data would have been better described by the mean direction $\bar{x}_o$ which is defined as

the direction of the resultant vector of a sample. Letting $\Theta_1, \Theta_2, \ldots \Theta_n$ be a sample of n directions, $\bar{x}_o$ is given by

$$\bar{X}_o = \arctan \frac{\sum_{i=1}^{n} \sin \Theta_i}{\sum_{i=1}^{n} \cos \Theta_i} \qquad (3)$$

This is simply a generalization of equation (2) to n vectors instead of two. In addition to being the direction of the resultant vector, $\bar{x}_o$ is the direction to the centroid of the observations $(\bar{C}, \bar{S})$. To see this, understand that the centre of gravity of a set of points has coordinates equal to the mean of the X coordinates and the mean of the Y coordinates; that is

$$\bar{C} = \frac{1}{n} \sum_{i=1}^{n} \cos \Theta_i \qquad (4)$$

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} \sin \Theta_i \qquad (5)$$

Because

$$\frac{\bar{S}}{\bar{C}} = \frac{\frac{1}{n}\sum_{i=1}^{n} \sin \Theta_i}{\frac{1}{n}\sum_{i=1}^{n} \cos \Theta_i} = \frac{\sum_{i=1}^{n} \sin \Theta_i}{\sum_{i=1}^{n} \cos \Theta_i}$$

it follows from (3) that the direction to the centroid must be the mean direction $\bar{x}_o$.

The mean direction possesses other interesting properties. For example, using the identities

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \qquad (6)$$

$$\cos(\alpha - \beta) = \cos\alpha \sin\beta + \sin\alpha \sin\beta \qquad (7)$$

it is not difficult to show that

$$\sum_{i=1}^{n} \sin(\Theta_i - \bar{x}) = 0 \qquad (8)$$

and

$$\sum_{i=1}^{n} \cos(\Theta_i - \bar{x}_o) = \text{maximum} \qquad (9)$$

The first property (equation 8), that deviates from the mean sum to zero, is similar to one possessed by the arithmetic mean. The second property (equation 9) states that of all possible angles, $\bar{x}_o$ is the one which causes the sums of cosines of deviations to be maximized. For this very reason (9) is useful in characterizing the dispersion or spread of a sample. In fact the maximum given by (9) is actually the *sample resultant length* R.. To see this use (7) to write

$$\sum_{i=1}^{n} \cos(\theta_i - \bar{x}_o) = \cos \bar{x}_o \sum_{i=1}^{n} \cos \theta_i + \sin \bar{x}_o \sum_{i=1}^{n} \sin \theta_i$$

From Figure 2 it is seen that

$$\cos \bar{x}_o = \frac{\Sigma \cos \theta_i}{R}, \quad \sin \bar{x}_o = \frac{\Sigma \sin \theta_i}{R}$$

thus

$$\sum_{i=1}^{n} \cos(\theta_i - \bar{x}_o) = \frac{1}{R} \left(\sum_{i=1}^{n} \cos \theta_i \right)^2 + \left(\sum_{i=1}^{n} \sin \theta_i \right)^2 = \frac{R^2}{R} = R$$

The resultant length R thus indicates the dispersion of a sample. For a group of n vectors with identical directions, R will equal n and a sample of perfectly opposing vectors would sum to zero. A large value of R therefore indicates a dense bundle of vectors with small spread, and conversely for small R. In practice R is divided by n to facilitate comparison between samples of different size - the statistic produced is called *mean resultant length* $\bar{R}$.

Circular variance is defined as

$$S_o = 1 - \bar{R} = 1 - \frac{R}{n} \qquad (10)$$

It is important to note that S_o is always between zero and one inclusive, making it very different from its linear analog, which can have any positive value.

It sometimes happens that observations have 'sense' or orientation but not direction. For example, one speaks of a road having a north-south orientation but a direction is not usually ascribed to the road in the way an automobile travelling north on the road is said to have direction. Data possessing orientation but not direction are called *axial data* or simply *orientation data*. In a study of pebble orientation in sedimentary deposits, Krumbein (1939) showed that orientations may be converted to directions simply by doubling the angles. This procedure yields the same angular measure for both components of the orientation.

Table 3 illustrates calculations of $\bar{x}_o$ and S_o for a set of directional data.

(ii) Other Measures

Angular observations are often grouped into classes or segments of the unit circle. The U.S. Weather Service, for example, divides the circle into eight classes of width 45° . A wind direction is reported according to the interval within which it lies. The calculation of statistics for grouped data is accomoplished by treating the observations of each class as if they all fell at the class midpoint. Equations (4) and (5) may be used to find C and S if the class frequency is multiplied by cos θ and sin θ before summation.

Table 3. Calculation of mean direction, x_o, and circular variance, S_o, for directions from Nairobi of mail destinations within Kenya (frequencies excluded)

i	θ_i	$\cos \theta_i$	$\sin \theta_i$	
1	17.25	.96	.30	
2	46.31	.69	.72	
3	47.60	.67	.74	
4	53.30	.60	.80	
5	54.16	.59	.81	
.	. .	.	.	
.	. .	.	.	
.	. .	.	.	
63	356.99	1.00	-.05	
		-2.26	18.04	Totals

$n = 63 \quad \bar{C} = -2.26/63 = -.0359 \quad \bar{S} = 18.04/63 = .286$

$\bar{R} = (\bar{C}^2 + \bar{S}^2)^{1/2} = (.0013 + .0818)^{1/2} = (.0831)^{1/2} = 0.288$

$\cos x_o = \bar{C}/\bar{R}_o = .125 \quad \sin x_o = \bar{S}/\bar{R} = .992$

$x_o = 7°11' \quad S_o = 1 - \bar{R} = 0.712$

Grouping a set of observations may bias an estimate slightly and to correct it perfectly it is necessary to apply a correction factor to statistics other than the mean which have been calculated from grouped data. The correction is small, however, if the class interval is not large and may usually be ignored as a matter of practice. For example, the correction to R is less than 3% for a class interval of 45 (Mardia, 1972: 39).

Other descriptive measures such as *median direction* and *circular mean deviation* for directional data are shown in Table 4. In most cases their calculation is straightforward and the meaning of each is comparable to that of its linear analog.

In closing this section we draw attention to the fact that the discussion thus far has been confined to unit vectors. That is, only the direction of the observations has been considered. All information concerning vector magnitudes is discarded when they are assigned unit length. For example, if wind speed and direction were measured only the direction 0 would appear in the formulae given above.

The formulae for $\bar{x}_o$ and R could easily be modified to include vector

TABLE 4. Descriptive Measures of Directional Statistics

Descriptive Measure	Formula	Definitions or Conditions
Mean direction	$\bar{x}_o = \arctan \frac{\sum_{i=1}^{n} \sin \Theta_i}{\sum_{i=1}^{n} \cos \Theta_i} + \lambda$	$\lambda = 0$ if $\sum_{i=1}^{n} \sin \Theta_i > 0, \sum_{i=1}^{n} \cos_i \Theta > 0$ $\lambda = \pi$ if $\sum_{i=1}^{n} \cos \Theta_i > 0$ $\lambda = 2\pi$ if $\sum_{i=1}^{n} \sin \Theta_i > 0, \sum_{i=1}^{n} \cos \Theta_i > 0$
Sample circular variance	$S_o = 1 - \frac{1}{n} \sum_{i=1}^{n} \cos (\Theta_i - \bar{x}_o)$	$\bar{x}_o$ is the mean direction
Sample circular standard deviation	$s_o = [-2 \log_e (1 - S_o)]^{\frac{1}{2}}$	$0 \le s_o < \infty$
Median direction	$\xi = \int_{\xi_o}^{\xi_o+\pi} f_\Theta \, d_\Theta = \int_{\xi_o+\pi}^{\xi+2\pi} f\Theta \, d\Theta = \frac{1}{2}$	$f(\xi_o) > (\xi_o + \pi)$
Mode direction	$\text{Mode} = L + \frac{f_o - f_{-1}}{2f_o - f_{-1} - f_{+1}} X h$	L = lower limit of the modal class f_o= frequency of the modal class f_{-1}, f_{+1}= frequencies of preceding and following modal classes h = length of class interval
Circ.Mean deviation	$d_o = \pi - \frac{1}{n} \sum_{i=1}^{n} \mid \{\pi - \mid\Theta_i - \alpha\mid \} \mid$	α = a fixed zero direction
Circ.Mean difference	$\Lambda_o = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i=1}^{n} \{\pi - \mid (\pi - \mid \Theta_i - \Theta_j \mid) \mid \}$	Θ_j = an angle other than Θ_i
Circular Range	$w = 2\pi - \max (T_1, \ldots T_n)$	$T_i = \Theta_{(i+1)} - \Theta_i$; $i=1,\ldots n-1$ $T_n = 2\pi - \Theta_n + \Theta_1$

magnitude by weighting each term in the summations according to an observation's length. The resulting values would have clear physical interpretation but statistical analysis as presented in the following sections would not be possible. It is not difficult to see why this is so. Just as is the case for linear statistics, directional statistical analysis is based on probability density functions (p.d.f. ^). A p.d.f. is a non-negative function whose integral, extended over the entire x-axis in linear statistics and from 0 to 360 in directional statistics, is unity. The random variable associated with a circular p.d.f. is an angle, and the function enables one to make statements about the probability of observing particular angles. As Steinmetz(1962) suggests, the use of vector lengths as weights involves postulating that the longer a vector is, the more indicative of direction it is. It is more likely that vector length will appear in a geographic study as a random variable in its own right. To consider direction and magnitude simultaneously would require a joint p.d.f.

We shall present only univariate distributions (indeed we know of no studies using linear-directional bivariate p.d.f. ^). Returning to the example of wind speed and direction, it could be said that linear statistics would enable one to investigate the marginal distribution of wind speed whereas the directional techniques presented here allow one to analyze the other marginal.

IV CIRCULAR PROBABILITY MODELS

A large number of circular probability models exists; like linear probability models, they may be either discrete or continuous. Several of the more important are discussed here.

Circular distribution functions and circular probability density functions are defined in a manner similar to their linear counterparts. Consider a random variable X which is valued only on the circumference of a unit circle. Any value of X can be associated with an angle Θ measured from an arbitrary reference direction. The distribution function, F, of X is defined by a probability, P_r, such that

$$F(\Theta) = P_r \ (0 < X \le \Theta), \quad 0 < \Theta \le 2\pi$$

Because

$$P_r \ (0 < X \le \Theta + 2\pi) - P_r \ (0 < X \le \Theta) = 1$$

we may write

$$F \ (\Theta + 2\pi) - F \ (\Theta) = 1, \ -\infty < \Theta < \infty$$

thus extending the domain of F beyond the interval $(0, 2\pi)$. A continuous distribution function F possesses a p.d.f., f, such that

$$\text{(i)} \int_{\Theta_1}^{\Theta_2} f(\Theta) d\Theta = F(\Theta_2) - F(\Theta_1), \quad -\infty < \Theta_1 < \Theta_2 < \infty, \quad |\Theta_1 - \Theta_2| < 2\pi$$

(ii) $f(\Theta) \geq 0, \ -\infty < \Theta < \infty$

(iii) $\int_0^{2\pi} f(\Theta) d\Theta = 1$

<u>(i) Uniform Distributions</u>

The simplest in form of all circular distributions is the uniform distribution. The continuous version of this distribution may be characterized by imagining a toy spinner which is as likely to come to rest in one direction as any other. Since there is no preferred direction the probability density is equal to a constant for all directions. From condition (iii) this constant must be $1/(2\pi)$ and the p.d.f. for a continuous uniform variable is

$$f(\Theta) = \frac{1}{2\pi}, \qquad 0 < \Theta \leq 2\pi$$

Suppose now that the movement of the spinner is restricted somehow so that the number of stopping points is reduced to some number m of equally spaced directions. If the probabilities associated with the stopping points are equal, the distribution remains uniform but is no longer continuous because probability is defined only at discrete points. Since the probabilities must sum to unity each direction has probability 1/m and the p.d.f. for a discrete uniform variable is

$$P_r\ (\Theta = v + 2\pi k/m) = 1/m, \quad k = 0,1,\ldots m-1$$

The shift parameter v appears above so that the zero direction need not be assigned a probability.

The discrete uniform distribution is actually a special case of a family of distributions called lattice distributions. Continuing with the spinner characterization, a general lattice distribution is produced if the requirement that all directions have equal probability is relaxed. The function continues to be defined only at m equally spaced points and the probability of the spinner stopping at the k^{th} direction is written P_k. The p.d.f. of the general lattice distribution is thus

$$P\ (\Theta = v + 2\pi k/m) = P_k, \qquad k = 0,1,\ldots,m-1$$

subject to the constraints

$$P_k \geq 0, \quad \sum_{m=0}^{m-1} P_k = 1$$

In the case where $P_0 = P_1, \ldots, P_{m-1} = 1/m$, a discrete uniform distribution obtains.

<u>(ii) The von Mises Distribution</u>

Probably the most important circular probability model is the von Mises distribution (von Mises, 1918) (see Figure 6). Used initially to investigate the clustering of atomic weights around integer values, the von Mises

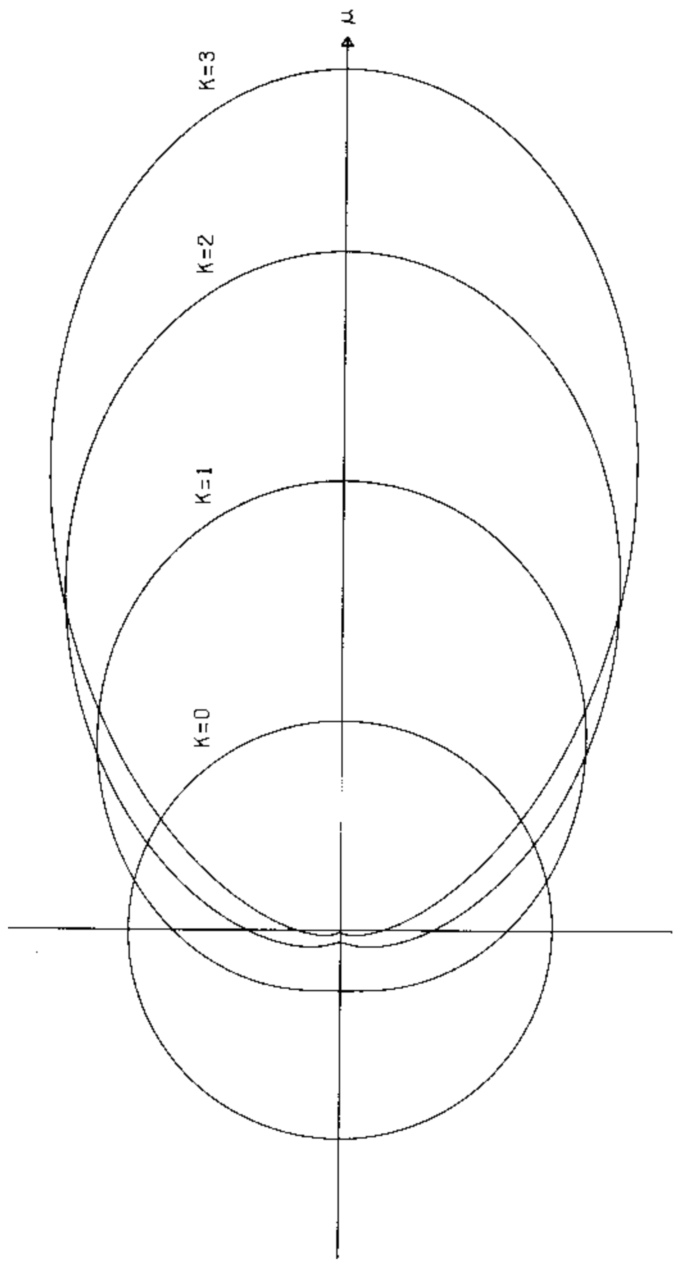

Figure 6. The von Mises distribution for different values of k.

distribution plays much the same role in circular statistical inference as the Gaussian distribution does in linear statistics. It has two parameters - a mean direction μ_o and a dispersion parameter k. Its p.d.f. is

$$f(\theta) = \left[2\pi I_o(k)\right]^{-1} \exp\left[k \cos(\theta - \mu_o)\right], \qquad (11)$$

$$0<\theta<2\pi,\ 0<\mu_o<2\pi,\ k>0$$

where

$$I_o(k) = \sum_{r=0}^{\infty} \left(\frac{1}{r!}\right)^2 \left(\tfrac{1}{2}k\right)^{2r}$$

The term $I_o(k)$ is a particular function of k called a modified Bessel function of the first kind and order zero; it has the effect of scaling the distribution so that (iii) is satisfied.

From (11) several important properties of the von Mises distribution can be deduced which are analogous to those of the Gaussian. Like the linear normal distribution it is completely determined by two parameters. The density is greatest when the argument of the cosine is zero, i.e., θ equal to μ_o. The mean direction is therefore also its mode, and by a similar argument the minimum occurs π radians away from μ_o. Because cos (-x) equals cos (x) the von Mises distribution (like the normal distribution) is symmetric about its mean.

The behaviour of (ii) under variation in k is also easy to see. As k approaches zero the exponential and Bessel functions approach unity and $f(\theta)$ approaches $\frac{1}{2\pi}$. The von Mises thus approaches a uniform distribution for small values of k. Alternatively, as k grows large the distribution becomes concentrated at μ_o (Figure 6). Because of these effects k is termed the concentration parameter and it is somewhat analogous to the reciprocal of the normal distribution's variance.

Because of its pre-eminence in hypothesis testing, its characteristics common to the linear normal, and because it can be constructed as a conditional distribution of the bivariate normal distribution, the von Mises is sometimes called the 'circular normal distribution'. This name is somewhat imprecise for when a true linear normal distribution is extended directly to the circle one obtains not a von Mises distribution but yet another circular distribution, the wrapped normal, which is important in the central limit theorem on the circle. In fact, the list of circular distributions is lengthened considerably when it is realized that any linear distribution with p.d.f. f(x) (continuous) or probability function p(x) (discrete) may be wrapped around the circle by

$$f(\theta) = \sum_{i=-\infty}^{\infty} (\theta + 2\pi i)$$

and

$$p(\theta = 2\pi k/m) = \sum_{i=-\infty}^{\infty} p(\theta + 2\pi i),\ k = 0,1,\ldots,m-1$$

respectively. Mardia (1972) discusses several wrapped distributions and

other circular distributions not mentioned here.

V HYPOTHESIS TESTING

A large number of inferential tests are available for variables distributed directionally, many of which are completely analogous to familiar linear tests. In most cases assumptions about the parent distribution must be made, but a number of non-parametric tests exist which may be used when assumptions about the population cannot be upheld. In this section, several geographical problems are analyzed to illustrate the variety and versatility of hypothesis-testing techniques available in directional statistics. Unless otherwise stated, the underlying population is assumed to approximate a von Mises distribution.

The procedure used in performing a test involving directional data is identical to that used in linear statistics. The researcher forms a null hypothesis which is tested against a mutually exclusive alternative hypothesis. The statistical theory on which the test is based allows one to reject the null hypothesis with a known probability of error. The probability of error associated with retaining an untrue null hypothesis depends upon the alternate hypothesis and is in general unknown. For this reason hypotheses are usually formed with the experimenter playing the role of a prosecutor trying to prove the null hypothesis untrue.

(i) Tests for Uniformity

The first hypothesis-testing technique demonstrated addresses the question of whether there is a directional bias to the data. Tests of this type aim to determine whether or not randomly sampled directions, $\theta_1, \ldots, \theta_n$, come from a population with a p.d.f. $f(\theta)$ such that θ is distributed uniformly over $(0, 2\pi)$, i.e.,

$$f(\theta) = (1/2\pi),\ 0<\theta\leq 2\pi$$

and there is no concentration around any particular direction.

The Problem: Is the pattern of interaction around Nairobi directionally biased? This type of problem is very common in spatial analyses. Tests for uniformity could be applied to hypotheses concerning von Thunen rings, Weberian transport costs, Christaller-type settlement patterns, or any hypothesis involving an isotropic plane.

The Data: The destinations within Kenya of all mail leaving Nairobi during one week were plotted. The directions of these destinations are the data and the sample size is 63. The directions were measured by overlaying a piece of transparent polar coordinate paper focussed on Nairobi but a protractor could also be used. If the data had not been mapped but had been available in Cartesian coordinates, the directions could have been determined by trigonometric calculations based on the definitions that the cosine of the angle is x/d and the sine of the angle is y/d where x,y are the Cartesian coordinates and d is distance, $d = (x^2 + y^2)^{\frac{1}{2}}$. The data for this problem are shown in Table 2.

The Hypothesis: Given a von Mises distribution the hypothesis of uniformity is

$$H_o:\ k = 0.$$

As was shown in Section 3, k is the concentration parameter of the von Mises distribution which is related (in a complex way) to circular variance. For k equal to zero f(0) degenerates to the uniform distribution. The alternate hypothesis is therefore

$$H_1 : k > 0$$

Table 5. Critical values* of the Rayleigh test of uniformity with the test-statistics $\bar{R}.Pr(\bar{R}>\bar{R}_o) = \alpha$

n α→	0.10	0.05	0.025	0.01	0.001
5	0.677	0.754	0.816	0.879	0.991
6	.618	.690	.753	.825	.940
7	.572	.642	.702	.771	.891
8	.535	.602	.660	.725	.847
9	.504	.569	.624	.687	.808
10	.478	.540	.594	.655	.775
11	.456	.516	.567	.627	.743
12	.437	.494	.544	.602	.716
13	.420	.475	.524	.580	.692
14	.405	.458	.505	.560	.669
15	.391	.443	.489	.542	.649
16	.379	.429	.474	.525	.630
17	.367	.417	.460	.510	.613
18	.357	.405	.447	.496	.597
19	.348	.394	.436	.484	.583
20	.339	.385	.425	.472	.569
21	.331	.375	.415	.461	.556
22	.323	.367	.405	.451	.544
23	.316	.359	.397	.441	.533
24	.309	.351	.389	.432	.522
25	.303	.344	.381	.423	.512
30	.277	.315	.348	.387	.470
35	.256	.292	.323	.359	.436
40	.240	.273	.302	.336	.409
45	.226	.257	.285	.318	.386
50	.214	.244	.270	.301	.367
100	.15	.17	.19	.21	.26
$2n\bar{R}^2 \sim \chi_2^2$	4.605	5.991	7.378	9.210	13.816

*Based on tables by Stephens and Batschelet with the kind permission of the authors.

Knowing (or assuming) the population to be von Mises allows the hypothesis of uniformity to be translated into an hypothesis about a parameter, k.

The Procedure and Results: The statistics $\bar{C}$, $\bar{S}$, and $\bar{R}$ are calculated as shown in Descriptive Measures section of this paper. A Rayleigh (1919) test for uniformity is applied by obtaining a critical value of $\bar{R}$ which is

asymptotically related to k, the Rayleigh test rejects the hypothesis of uniformity for large values of $\bar{R}$. Interpolating from Table 5, a critical value of 0.2225 is obtained. Since in this example $\bar{R}$ = 0.2888 (see Table 3), the hypothesis of uniformity is rejected. There is evidence of directional bias in the interactance pattern around Nairobi. The test outlined above assumes the population mean direction μ_o to be unknown. In the less frequent case where the mean direction is known and is adjusted to zero, the Rayleigh test is inappropriate. In this case the test statistic is the mean of the cosine of the angles, $\bar{C}$, and the critical values of $\bar{C}$ may be found in Table 6. $\bar{C}$ is simply calculated by:

$$\bar{C} = C/n$$

where C is the sum of the cosines of the angles (see Descriptive Measures section) and n is the size of the sample. A more detailed description of this test can be found in Stephens (1960).

Table 6. Critical values* of the test statistic $\bar{C}$ for testing uniformity when the mean direction is known. $Pr(C \geq C_o) = \alpha$:

n α→	0.10	0.05	0.025	0.01
5	0.413	0.522	0.611	0.709
6	.376	.476	.560	.652
7	.347	.441	.519	.607
8	.324	.412	.486	.569
9	.305	.388	.459	.538
10	.289	.368	.436	.512
11	.275	.351	.416	.489
12	.264	.336	.398	.468
13	.253	.323	.383	.451
14	.244	.311	.369	.435
15	.235	.301	.357	.420
16	.228	.291	.345	.407
17	.221	.282	.335	.395
18	.215	.274	.326	.384
19	.209	.267	.317	.374
20	.204	.260	.309	.365
21	.199	.254	.302	.356
22	.194	.248	.295	.348
23	.190	.243	.288	.341
24	.186	.238	.282	.334
25	.182	.233	.277	.327
30	.17	.21	.25	.30
35	.15	.20	.23	.28
40	.14	.18	.22	.26
45	.14	.17	.21	.25
50	.13	.16	.20	.23
$(2n)^{\frac{1}{2}}\bar{C} \approx N(0,1)$	1.282	1.645	1.960	2.326

*Based on a table by Stephens with the kind permission of the author.

(ii) One Sample Test for Mean Direction

This test asks whether the directional bias of the data clusters around a specified direction. It is assumed that randomly sampled directions, $\theta_1,\ldots,\theta_n$, have a circular variance less than unity (i.e., they are from nonuniform distributions) and that the population has a von Mises distribution with mean μ.

The Problem: Kenya underwent spatial reorganization under the colonialism of the British. The initial spatial reorganization of the country's interior was dominated by a railway built by the British to link Uganda with the Kenyan port of Mombasa. In 1889, the railhead reached Nairobi, an undeveloped site suitable as a base for further railway construction. Kenya and Nairobi have since grown rapidly. Has the spatial organization of interactance in contemporary Kenya changed from its initial orientation to the railway axis?

The Data: The data are the same as in Example 1.

The Hypothesis:

H_o: $2\mu_o = 272^0$ (the doubled orientation of the railway)

In this case, the null hypothesis is that the orientation mean of the Kenyan interactance pattern (as measured by mail destinations) is the same as the orientation of the railway. It is assumed that the 63 angles sampled come from a von Mises distribution and we expect the sample orientation mean should be reasonably close to the population orientation mean if the null hypothesis is plausible. The inferential power of this statistical technique lies in the fact that confidence intervals can be constructed (analogous to those constructed utilizing the standard deviation in linear statistics) to determine whether the sample orientation mean falls within the bounds which take into account the variance which can be expected using a sample taken from a von Mises-distributed population. If the sample mean falls outside the confidence interval, we can say with a specified degree of confidence that the sample orientation mean is not the same as the orientation of the railway.

The Procedure and Results: Given that the test is for orientation, all angles are doubled (see Table 7). The sample mean, $\bar{x}_o$, is 246^0 and the mean resultant length, $\bar{R}$ is 0.14145. The size of the confidence interval about $\bar{x}_o$ depends in part upon the population variance. Since k is unknown R is used in determining the interval with δ (Figure 7). Since the concentration parameter for the population is unknown the test procedure utilizes one of Batschelet's (1971) charts to obtain a confidence interval around the sample mean (see Figure 7). As n = 63 and R = 0.4145, a 95% confidence interval of 246 ± 27 is found. Since the hypothesized mean of 272 falls within this confident interval, the null hypothesis cannot be rejected at the 5% level of significance. The contemporary mean direction of interactance in Kenya is not significantly different from the orientation of the railway axis.

Alternative one sample tests for mean direction have been developed to test similar hypotheses. One set of these tests covers the infrequent cases when the concentration parameter, κ, is known. If $\kappa>2$, the test is based on a Neyman-Pearson approach using Stephen's (1969a) approximation to obtain the test statistic. If κ is small, the Fisher (1959) ancillary principle allows testing which only considers the distribution of $\bar{x}_o|R$. These tests depend on μ_o. Mardia (1972) has derived a conditional unbiased test against the composite hypothesis $\mu \neq 0$. The above alternative

Table 7. Orientation data of mail sent from Nairobi during one week for destinations within Kenya. The angles of these data have been doubled to test for orientation.

Case	Orientation Angle	Case	Orientation Angle	Case	Orientation Angle
1	34.50	22	243.66	43	325.06
2	92.62	23	252.94	44	330.76
3	95.20	24	253.26	45	85.66
4	106.60	25	254.24	46	170.00
5	108.32	26	255.74	47	170.20
6	108.58	27	257.98	48	227.42
7	122.62	28	267.42	49	236.60
8	125.64	29	271.36	50	237.00
9	129.50	30	273.70	51	241.30
10	130.28	31	275.18	52	245.60
11	134.76	32	276.20	53	251.08
12	147.32	33	277.70	54	256.32
13	160.62	34	282.24	55	258.88
14	161.08	35	285.88	56	264.84
15	163.06	36	292.26	57	266.34
16	167.00	37	296.38	58	268.46
17	170.78	38	301.56	59	269.16
18	212.28	39	308.68	60	270.62
19	213.40	40	316.40	61	298.68
20	217.94	41	320.26	62	301.90
21	226.90	42	324.38	63	353.98

tests and other alternative tests for mean direction when the concentration parameter is unknown, including one-sided tests, are detailed in Mardia (1972).

(iii) Two Sample Tests for Mean Direction

This test addresses the question of whether samples from two populations have the same mean direction. Tests of this type assume that the concentration parameters of the two populations, k1 and k2 are equal and that the populations have von Mises distributions with means p and μ_2

The Problem: That the seasonality of precipitation can be analyzed by the use of vector lengths has been shown by Markham (1970), but because the vector lengths used were proportional to the magnitude of rainfall rather then frequency, it was not possible to perform statistical tests based on frequency distributions. To demonstrate an alternative method that allows for inferential testing, the question is asked, Is there a difference in the seasonality of rainfall for San Francisco and Tucson?"

The Data: Both San Francisco and Tucson received just over 12 inches of precipitation in 1971 and it rained approximately the same number of days in both places. The number of days it rained in San Francisco and Tucson in 1961 are shown in Table 8.

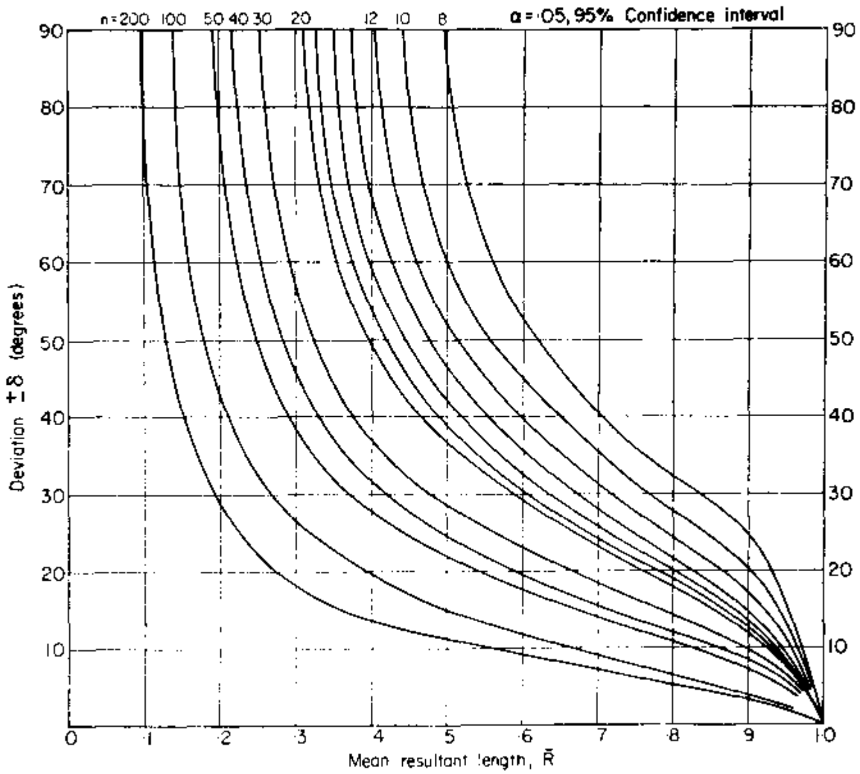

Figure 7 Confidence intervals for μ_o. Batschelet chart for obtaining a 95% confidence interval for the mean direction μ_o
$Pr\ (-\delta<\mu_o<\delta) = 0.95$

Reproduced from Batschelet (1971) with the kind permission of of the author and the publisher, American Institute of Biological Sciences.

The Hypothesis:

H_o: $\mu_1 = \mu_2$ The mean direction and thus 'seasonality' of the two populations are equal. In this hypothesis, the sample means are compared statistically in order to determine whether it is likely that they came from populations which have the same population mean. In this example, the population directional means correspond to a specific time of year - the peak rainfall season.

The Procedure and Results: The use of directional statistics is not limited to the analysis of directions, but is applicable to many other cyclic variables. When time-periodic data are analyzed, the time period (day, month, year, etc.) is associated with a 360 cycle and observations at a particular time are assigned an angular value corresponding to the proportion

Table 8 A Modification of Time-Periodic Data to Directional Analysis
The Number of Days of Rainfall in Tuscon and San Francisco in 1971.

Sample 1 (San Francisco)

Month	Θ	f_i observed freq.	$f_i(30/D_i)$	$f_Θ$ adjusted freq.
Jan	0	8	7.74	7.85
Feb	30	5	5.36	5.43
Mar	60	5	4.84	4.91
Apr	90	6	6.00	6.08
May	120	3	2.90	2.94
Jun	150	0	0.00	0.00
Jul	180	0	0.00	0.00
Aug	210	1	0.97	0.98
Sep	240	2	2.00	2.03
Oct	270	3	2.90	2.94
Nov	300	7	7.00	7.00
Dec	330	14	13.55	13.74
		N=54	N'=53.26	54.00

Sample 2 (Tucson)

Month	Θ	f_i observed freq.	f_i $(30/D_i)$	$f_Θ$ adjusted freq.
Jan	0	1	0.97	0.99
Feb	30	3	3.21	3.28
Mar	60	0	0.00	0.00
Apr	90	2	2.00	2.04
May	120	1	0.97	0.99
Jun	150	0	0.00	0.00
Jul	180	8	7.74	7.90
Aug	210	15	14.52	14.82
Sep	240	5	5.00	5.10
Oct	270	7	6.77	6.91
Nov	300	2	2.00	2..04
Dec	330	6	5.81	5.93
		N=50	N'=48.99	50.00

Table 8 (continued)

Combined Sample

Month	Θ	f_i observed freq.	f_i (30/D)	f_Θ adjusted freq.
Jan	0	9	8.71	8.86
Feb	30	8	8.57	8.72
Mar	60	5	4.84	4.92
Apr	90	8	8.00	8.14
May	120	4	3.87	3.94
Jun	150	0	0.00	0.00
Jul	180	8	7.74	7.87
Aug	210	16	15.48	15.75
Sep	240	7	7.00	7.12
Oct	270	10	9.68	9.84
Nov	300	9	9.00	9.15
Dec	330	20	10.35	19.68
		N=104	N'.102.25	104.00

of time period elapsed before the observation. Equation (12) represents this symbolically where P is the period, t_i is an observation at time t, and Θ_i is the same point expressed as an angle:

$$\Theta_i = (t_i/P) \cdot 360. \quad (12)$$

When the data are grouped by month, the frequencies must be adjusted to account for the inequality of class intervals. This can be accomplished by creating a new year of 360 days having twelve months of thirty days each. The adjusted frequencies are found by:

$$f_\Theta = f_i \ (30/D_i) \ (N/N')$$

where f_i is the observed frequency of month i, D_i is the number of days in month i, N is the sample size, and

$$N' = \sum_{i=1}^{12} f_i \ (30/D_i)$$

Using the adjusted frequencies (see Table 8), Watson and Williams' (1956) two sample test can be used to test the null hypothesis. The Watson and Williams' test is conditional only on R and does not depend on $\bar{x}_o$, which contrasts it with other similar tests which are conditional on $\bar{x}_o$ and R.

Since for a fixed resultant of the combined sample (R), the sum of the

individual sample resultant lengths (R_1, R_2) increases as their directions diverge, H_o is rejected for large values of $R_1 + R_2$. When $\bar{R}$ is close to unity (>0.7) the critical values of the F-distribution can be used according to equation (13):

$$F_{1,n-2} = (1 + 3/8k)\ (n-2)\ (R_1 + R_2 - R)/(n - R_1 - R_2) \quad (13)$$

where k is estimated by the use of Gumbel's (1954) Table 2 or Batschelet's (1971) Table B.

Since R_o = 0.241 (and is thus not close to unity) for this example, a critical 5% value for Watson and Williams' two-sample test of R' of 0.27 where $R' = (R_1 + R_2)/n$ is alternatively found in Mardia's (1972) Appendix 2.9a. For the San Francisco-Tucson example, the test statistic R' = 0.51, thus the hypothesis of equal mean directions is strongly rejected; it is concluded that a seasonal difference exists in the precipitation patterns of the two locations.

(iv) Two Sample Tests for Concentration Parameter

This test addresses the question of whether the circular variances of the samples from two populations are the same. Tests of this type utilize an approximation to the distribution of $\bar{R}$. The population is assumed to have a von Mises distribution.

The Problem: The analysis of river meanders is an important aspect of fluvial geomorphology which has not been wholly integrated into the theory of river systems. Most sinuosity indices (Mueller, 1968) of river meanders do not measure the fundamental aspect of meander-changes in direction. It is proposed here that the circular variance be used as a testable index of stream meanders. For example, is the meander pattern of a plains river (the Little Missouri) different from the meander pattern of a mountain river (the Klamath)?

The Data: From the U.S.G.S. 1:24000 maps of the Klamath and Little Missouri rivers, a starting point and fifty sample points at 1000 feet intervals upstream were chosen for each river. Using a method similar to that of Thakur and Scheidegger (1968), each midstream point was linked to its neighbour by astraight line whose direction was found relative to the direction of the main valley axis.

The Hypothesis:

H_o : $k_1 = k_2 = k$. In this hypothesis, k is unknown. Since the tests are to determine the equality of the variance of the two samples, they do not depend on $\bar{x}_o$.

The Procedure and Results: The choice of test statistic in this case depends upon the value of $\bar{R}$ as detailed by Mardia (1972):

Case 1) $\bar{R}$<0.45

The statistic G is approximately normally distributed with mean zero and variance unity:

$$G = \frac{2}{\sqrt{3}} \left| \sin^{-1}(1.22474\ \bar{R}_1) - \sin^{-1}(1.22474\ \bar{R}_2) \right| / \left| (n_1-4)^{-1} + (n_2-4)^{-1} \right|^{\frac{1}{2}}$$

Case 2) $0.45 \leq \bar{R} \leq 0.70$

The statistic G is again N(0,1):

$$G = |g(\bar{R}_1) - g(\bar{R}_2)| \quad 0.89325 \quad |(n_1 - 3)^{-1} + (n_2 - 3)^{-1}|^{-\frac{1}{2}}$$

where

$$g(R) = \sinh^{-1} |(R - 1.0894) / 0.25789|$$

Case 3) $0.70<\bar{R}$

G now has an F distribution with n_1-1 and n_2-1 degrees of freedom:

$$G = |(n_1 - R_1)(n_2 - 1)| / |(n_2 - R_2)(n_1 - 1)|$$

For this example $\bar{R} = 0.5373$ and, since $0.45\leq\bar{R}\leq70$, and

$g(R_1) = 1.805$; $g(R_2) = 1.095$

thus G = 3.853 which is greater than a critical value at the 95% level of 1.96 and the null hypothesis is rejected. The two rivers have different meander patterns.

(v) Non-Parametric Tests

Except for the test for uniformity, the tests presented thus far require the assumption that the underlying population can be represented by a von Mises distribution. In directional statistics, as in linear statistics, situations arise in which such assumptions about the form of the parent population are untenable. Fortunately, a variety of non-parametric directional tests exists which affords a fair amount of flexibility to the researcher.

The Problem: It has been suggested that the 'ritual custom of *agni mull* which prohibits the taking of a wife from the southeast or northwest of one's own village' has affected mate selection in Mysoreian villages (McCormack, 1958). Could directional statistics be used to test the adherence to this cultural taboo?

The Data: Table 9 presents hypothetical responses from 17 villagers to the question: 'Which direction would you prefer your wife to come from?'

The Hypothesis:

H_o: $F(\Theta) = \Theta/2\pi$. For a discussion of $F(\Theta)$ see the circular probability models section.

The Procedure and Results: Single sample tests usually fall into one of two categories tests for uniformity and tests for symmetry. In testing the hypothesis that a distribution is symmetric about a given direction, it is possible to use symmetry tests adapted from linear statistics. Schach (1969) has outlined the use of the Wilcoxon test for circular data and discussed its efficiency relative to tests for von Mises populations. The sign test was evaluated in a similar manner and found to be more powerful than the Wilcoxon test for small k and large n.

There are several non-parametric tests for uniformity of circular data. Among the more important are Kuiper's (1960) V-test, Watson's (1961) U^2-test, and Ajne's (1968) A_n-test. The latter two are special cases in a general class of uniformity tests discussed in considerable detail by Beran (1969).

Table 9 Hypothetical Preferred Directions of Prospective Brides by 17 Mysoreian Villagers

	Θi	Θi/360	i/n	Ui-i/n
1	40	.111	.059	.052
2	54	.150	.118	.032
3	73	.203	.176	.027
4	85	.236	.235	.001
5	92	.256	.294	-.038
6	114	.317	.353	-.036
7	154	.428	.412	.016
8	170	.472	.471	.001
9	178	.494	.529	-.035
10	191	.531	.588	-.057
11	220	.611	.647	-.036
12	248	.689	.706	-.017
13	280	.778	.765	.013
14	293	.814	.824	-.010
15	307	.853	.882	-.029
16	321	.892	.941	-.049
17	346	.961	1.000	-.039

Stephens (1969b) has compared the relative power of the three tests using Monte Carlo trials. Although the power of a test varies with the alternative considered, in general it can be said that Kuiper's V-test is preferred, particularly for small samples.

Kuiper's test is similar to the Kolmogorov-Smirnov test in that it is based on the maximum deviations of the observed distribution from the expected. Following Mardia (1972, ch.7), a distribution function of fixed zero direction is defined by:

$$S_n(\Theta) = i/n \quad \text{if } \Theta_i \leq \Theta \leq \Theta_{(i+1)}, \quad \text{for } i=0,1,2,\ldots,n$$

and

$$\Theta_{(0)} = 0; \qquad \Theta_{(n+1)} = 2\pi$$

The expected value of $S_n(\Theta)$ is $F(\Theta)$ and Kuiper's statistic, distribution free and invariant under rotation, is based on the maximum differences between the two. It is given by:

$$V_n = \max_{1\leq i\leq n} (U_i - i/n) - \min_{[1\leq i\leq n]} (U_i - i/n) + 1/n$$

where $U_i = \Theta_{(i)}/2\pi$. Stephens (1965) has tabled the upper percentage values of $n^{\frac{1}{2}}V_n$. The hypothesis of uniformity is rejected for large values.

The hypothetical observations from Table 9 are placed in order and

$$n^{\frac{1}{2}}V_n = \left[4.12 \quad 0.52 - (-0.057) + 1/17\right] = 0.892$$

Testing at the 5% level, the critical value is 1.66 for n=17 and thus the null hypothesis is not rejected. The cultural taboo is not validated by the data.

VI SPHERICAL PROBABILITY DISTRIBUTIONS

Thus far the discussion has been limited to the analysis of vectors in the same plane with a single angle specifying a unique position within the distribution. When directions in free space are of interest, a circular distribution will no longer suffice- it is necessary to consider the distribution to exist on the surface of a sphere. Vectors in space may occur, for example, in studies of bedding plane strike and dip, paleocurrent orientation, plant preferences of slope angle and azimuth, and occurrences of phenomena at the world scale. As geographers know, the representation of spherical data in only two dimensions requires a projection. Watson (1970) discusses those projections which are appropriate for angular data.

The direction of a vector in three dimensions can be specified in several ways. The method most common to geographers is the specification of latitude and longitude. If latitude is replaced by the angle the vector makes with the positive z-axis (Θ), the spherical coordinate system frequently used in analytic geometry results. The angle just described (Θ) varies between 0 at the north pole and π. The x- and y-axes are found in the plane of the equator ($\Theta=\pi/2$) and the longitude ϕ is measured counterclockwise from the positive x-axis (Figure 8).

A vector may also be specified by direction cosines which correspond to its components along the three coordinate axes. If α, β, and δ are the angles the vector makes with the x-, y-, and z-axes, then the direction cosines are given by

$$a = \cos\alpha, \quad b = \cos\beta, \quad c = \cos\delta.$$

Direction cosines are related to spherical coordinates by

$$a = \sin\Theta\cos\phi, \quad b = \sin\Theta\sin\phi, \quad c = \cos\Theta \quad 0\leq\Theta\leq\pi, \quad 0\leq\phi<2\pi.$$

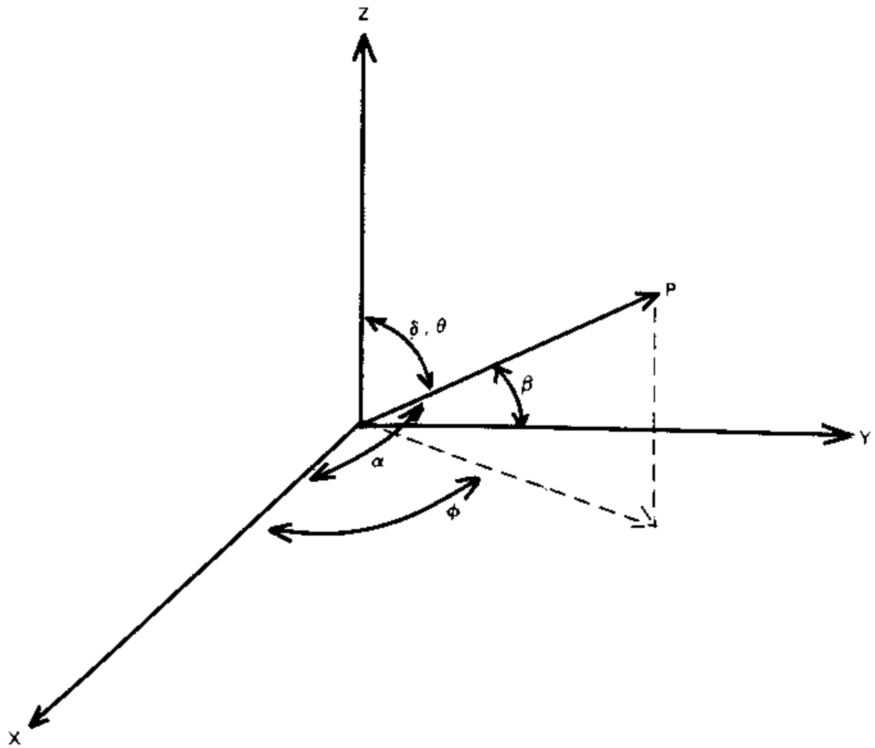

Figure 8 Measuring angles on a sphere.

The direction cosines lead to the spherical mean direction of a group of vectors which is defined similarly to the circular mean direction--as the direction of the resultant vector. If R is the length of the resultant. the direction cosines of the resultant (a_o, b_o, c_o) are found by

$$a_o = \sum_{i=1}^{n} (a_i/R), \quad b_o = \sum_{i=1}^{n} (b_i/R), \quad c_o = \sum_{i=1}^{n} (c_i/R)$$

where

$$R = \left[\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 + \left(\sum_{i=1}^{n} c_i\right)^2\right]^{\frac{1}{2}}$$

R is again a measure of concentration and the spherical variance is defined by

$$S_o = 1 - R/n, \quad 0\leq S_o\leq 1.$$

Distribution theory for the sphere is considerably more complex than for the circle and as a result the number of inferential tests is presently somewhat restricted. The circular uniform and von Mises distributions have been extended to the sphere, however, and the p.d.f.s of these and several other spherical models have led to tractable sampling distributions. Mardia (1972) has summarized recent work and has provided several charts and tables useful for hypothesis testing. A spherical test for uniformity is illustrated below.

(i) Spherical Test for Uniformity

The Problem: Some plants prefer slopes of particlar inclination and/or aspect for edaphic, climatic, or phenological reasons. Are plants of the genus *Lupinus* distributed in a uniform manner with respect to slope angle and azimuth?

The Data: The data were collected as part of a larger study by John F. O'Leary, 'Habitat Preferences of the Genus *Lupinus* in a Portion of the Western Transverse Ranges of Southern California,' unpublished Master's thesis, Dept. of Geography, U.C.L.A. Eight transects, one from each sector defined by cardinal direction, were randomly selected from a larger sample of transects in Los Angeles and Ventura counties along which members of the genus *Lupinus* were found. Along each transect (which was aligned perpendicular to slope contour) the presence or absence of lupines in contiguous one square metre quadrats was noted. For those quadrats in which lupines were found, the slope angles were recorded, which, together with the azimuth of the quadrat formed a paired observation (S,A) of slope angle and direction. These observations are mapped on the sphere by letting $\Theta = 90 - S$, $\phi = A$. Because ϕ (azimuth) varies between 0 and 2π, Θ is restricted to angles of not more than $\pi/2$ (potential overhanging slopes are excluded) and the distribution is seen to exist on an upper hemisphere only.

Hypothesis: We wish to test the hypothesis that the sample vectors are uniformly distributed against the alternative that they arise from a rather complicated distribution investigated by Bingham (1964). The shape of his distribution is controlled by three parameters k_1, k_2, k_3 which determine regions of concentration. When these are equal the distribution becomes uniform thus our hypothesis is

$$H_o: k_1 = k_2 = K_3$$

The Procedure and Results:

Watson has shown that the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of the matrix of the sums of squares and products T which are related to k_1, k_2, k_3 are useful diagnostic aids to determining the general shape of a spherical distribution. (The use of eigenvalues and eigenvectors assumes a knowledge of matrix algebra. Readers unfamiliar with these concepts should consult a text on matrix algebra.)

$$T = \begin{bmatrix} \Sigma a_i^2 & \Sigma a_i b_i & \Sigma a_i c_i \\ \Sigma a_i b_i & \Sigma b_i^2 & \Sigma b_i c_i \\ \Sigma a_i c_i & \Sigma b_i c_i & \Sigma c_i^2 \end{bmatrix}$$

He has shown that under the hypothesis of uniformity $\lambda_1 = \lambda_2 = \lambda_3 = (n/3)$ since $\Sigma\lambda_i = n$.

The test statistic used here is based on the differences $\lambda_i - (n/3)$ and is distributed as chi-square with five degrees of freedom. Valid only for large n, it is given by

$$S_u = (15/2n) \sum_{i=1}^{n} [\lambda_i - (n/3)]^2.$$

For the lupine data the following values obtain:

$$\sum_{i=1}^{n} a_i = 34.93, \quad \sum_{i=1}^{n} b_i = 12.96, \quad \sum_{i=1}^{n} c_i = 34.30, \quad R = 50.65.$$

These imply a mean direction of (43.340) (note that the mean direction does not suggest the average slope was 43 -- the resultant necessarily points to the center of gravity of the distribution). The T matrix is:

$$\begin{bmatrix} 117.85 & -9.64 & 1.83 \\ -9.64 & 43.86 & 3.93 \\ 1.83 & 3.93 & 9.28 \end{bmatrix}$$

with eigenvalues of 8.76, 43.13 and 119.1. S_u for the sample is approximately 280, thus the hypothesis of uniformity is strongly rejected as X^2_5 at the 1% level is only 15.1. It is concluded that the lupines exhibited slope-aspect preferences.

VII GEOGRAPHICAL APPLICATIONS

Most current spatial work ignores directional influence or assumes circular concentricity. Indeed, there is a sphere of influence in geography reminiscent of the influence in cosmology of Plato's 'circular dogma', i.e., the shape of the world must be a perfect sphere and all astronomical motion must be in perfect circles at uniform speed. Koestler claims that through this circular dogma, 'Plato laid a curse on astronomy whose effects were to last till the beginning of the seventeenth century' (1959, p.60). A consideration of the circular-based major theories in geography (e.g., Weber, von Thunen, Christaller) suggests a disconcerting parallel.

The geographic literature reflects virtual neglect of applications of directional statistics. Notable exceptions are the studies by Clark (1972); Gould (1967); K.E. Haynes and W.T. Enders (1975); and Cadwallader (1977). An example of a study that could have been rendered more precise with the use of directional statistics is J. Adams (1969). The lacuna appears more profound when possible applications to geography are considered. In the above examples, it is intended not just to demonstrate directional statistical techniques, but also to illustrate some of their possible applications to geographic research. Other applications are considered below.

Mathematical geographers should investigate directional statistics as a field in its own right within the sphere of geomathematics. The techniques may have further applications to the study of shape, pattern, and orientation. The quantitative analysis of shape is dominated by indices calculated from distances (measured along radials, axes, or perimeters) and often requiring the index to be relative to geometric shapes or shape such as a circle or polygon. It might be possible to use Taylor's (1971) method of overlaying a rectangular grid of points on the shape, and, rather than measuring the distance from each circumscribed point to all other circumscribed points, measure the direction (irrespective of initial orientation). The circular variance of the measured directions may be a useful alternative measure of shape and has the additional asset of being easily tested inferentially. Pattern has been analyzed through quadrat analysis, Lexis numbers, and near-neighbor analysis - all distance-based statistics (with quasi-directional variation by Dacey and Tung, 1962). Only recently have directional characteristics of point patterns been analyzed (Haynes and Enders, 1975; Boots, 1974). The quantitative assessment of orientation, often attempted obliquely (Blair and Bliss, 1967), is definitively resolved with the use of directional statistics. Should geomathematicians manage to transfer the techniques of circular regression and vector trend surface analysis from exploratory to operational status, the range of possible applications throughout the spatial sciences would be considerably increased.

Urban-economic geographers will find directional statistics complementary to their existing base of structural research as well as to more contemporary dynamic concerns. Transport networks have definite directional attributes that have been woefully neglected in research predominantly utilizing graph-theoretic methodology. Using frequencies to depict magnitudes and changes of flows, directional statistics can accomodate dynamic analyses of transport.

The dynamic notions of changing urban forms have been voiced in a variety of speculations from the historical evolution of Gottmann's (1961) 'Megalopolis' through Friedmann and Miller's (1965) more general 'urban fields'. Are cities represented by their delineated forms growing towards each other (is there evidence for urban 'gravity')? Directional statistics could test whether the growth of urban form is directionally biased towards other urban areas. (A pilot study by one of the authors did not support this hypothesis.) Directional statistics can be applied to central place studies. Mardia (1972) discusses theoretical lattice distributions that may be well-suited to central place analysis and particularly to the Losch (1954) variant.

The urban-structural debate of Hoyt's (1933) sectors versus Burgess's (1925) rings has continued for decades. Comparative analysis of the two models has often been inconclusive, although Yeates (1965) was able to demonstrate the importance of direction on land value prediction by analysis along individual sectors. Directional statistics could assist in the clarification of this dilemma. The generalized land surface values within a city posited by Berry et al. (1963) can be seen to have definite directional attributes. Studies of urban-hinterland development surfaces have also isolated strong directional components.

Both Weber's and von Thunen's models are based on transport costs which vary radically with respect to distance, depending on the existence of transport arteries (von Thunen explicitly recognized this), yet the models are most usually employed incorporating an assumption of concentric

transport costs. Hamilton (1967), in his concentric model of the industrial structure of a metropolis, notes 'the 'wedging' effect of transport lines on the distribution pattern'. The explicit incorporation of direction into these and similar models may be detrimental to their elegance, but not to their utility.

Human geographers will find directional statistics an aid to their analyses of diffusion. Agricultural dispersions usually display a preferred direction relative to their origin. Diffusion models are often concentric ('wave-like') or hierarchic (essentially aspatial), but seldom directional. The mean information fields of Monte Carlo simulation of diffusion may be modified to employ a polar-coordinate field, with probability values varying in relation to theoretically- or empirically-derived mean direction and circular variance. An example of the directional bias of diffusion can be seen in the analysis of Kenya mail data in Example 1 and 2. Cadwallader (1977) has used directional statistics to determine directional biases (frame dependency) in cognitive maps.

Biogeographers and mathematical ecologists will find a variety of uses for the techniques. Organisms not only diffuse non-randomly, but also with preferred direction. Zoogeographers will find that zoologists and ethologists have already applied directional statistics to the migration and orientation behaviour of ducks, turtles, and homing pigeons- the latter study offering the less-then-startling conclusion that homing pigeons fly home. Plant geographers should note the application of the spherical (three-dimensional) techniques to the relationship between plant distributions and physical features shown in Example 6.

The use of directional statistics has been more widespread in studies allied to physical geography. The most common application has been in the study of bedding plane and grain axis orientation with the majority of references appearing in the geologic rather than geographic literature (e.g., Scheidegger, 1965; Williams, 1972; Mark, 1973). Steinmetz(1962) has tabled some of the uses to which directional statistics have been put; other reviews may be found in Jones (1968) and Pincus (1956).

Since many climatological processes are time-periodic (e.g., Example 3), they are amenable to analysis using directional techniques. Other climatological applications include the study of wind directions which has been preliminarily explored by meteorologists. Their initial research indicates that the distributions of wind directions can be closely approximated by a particular bivariate circular probability density function, but much inferential work remains to be done using this model.

Extending his wind power studies, Hardy (1977) has developed a new way to analyze vector magnitude and direction simultaneously. Each observation vector is associated with a complex number $se^{i\theta}$ where s is a vector magnitude and i equals $\sqrt{-1}$. A Hermitian matrix is then formed and diagonalized as in ordinary principal components analysis. In the case of Hardy's work this has resulted in the delineation of regional wind patterns. The technique has obvious potential application to other vector fields and its relationship to the techniques 'presented here should be examined.

VIII CONCLUSION

Directional statistics have been introduced as a set of techniques with considerable potential application to geographical research. Given that direction is an explicitly spatial concept, the adoption of these techniques by geographers will facilitate the inclusion of this geographical primitive into their analyses. As with linear statistics, an understanding of theory and models underlying directional statistics requires considerable mastery of mathematics, yet the simple application of the variety of mathematically acceptable tests demonstrated in the examples of this work is not predicated on such mastery.

The fact that geographers have not rigorously included direction in their research is understandable given the dominant use of linear statistics in science. That a linear rather than a directional theory of statistics developed is ironical given the genesis of the arithmetic mean and the theory of errors:

> Indeed the theory of errors was developed by Gauss primarily to analyze certain directional measurements in astronomy. It is a historical accident that the observational errors were sufficiently small to allow Gauss to make a linear approximation and, as a result, he developed a linear rather than a directional theory of errors (Mardia, 1972, p.xvii).

It is recommended that geographers be among the first to redirect their approach to quantitative analyses from the straight and narrow path of historical accident.

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