LINEAR PROGRAMMING: THE SIMPLEX METHOD WITH GEOGRAPHICAL APPLICATIONS

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by

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CONTENTS

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I INTRODUCTION

(i) Introductory Comments

Consider the following problems. The first concerns a farmer who has at his disposal given amounts of land, labour and water (for irrigation purposes). He may produce two crops. Each ton of Crop 1 produced yields a certain known profit but requires that certain known amounts of the three resources be used up. Each ton of Crop 2 produced also yields a known profit and involves using known amounts of the three resources. An obvious question which arises is: how much of each of the crops should the farmer produce in order to maximise his profit while not using more resources than are available?

The second problem relates to a situation which might be faced by the Deliveries Manager of a large company. Known amounts of a particular commodity are produced at a set of factories each month. The commodity is to be delivered to a set of retail outlets each of which requires a known amount of it per month in order to meet customers' needs. The cost of transporting a given quantity (say one ton) of the commodity from a factory to a retail outlet varies as it depends, among other things, on how far apart the factory and retail outlet are. This means that one factory/outlet delivery pattern will probably involve a different transport cost to another which raises the question: what pattern of factory/retail outlet deliveries minimises total transportation costs while ensuring that no factory is called upon to deliver more than it can produce and each retail outlet receives its requirement?

The final problem represents a type of situation often confronted at governmental level. A given sum of money is available for investment in development projects. A large number of projects on which some of the money could be spent has been proposed. Each project will yield a given return. Thus to spend the money on one set of projects would yield a return different to that accruing if a different set is chosen. This raises the question of which particular set of projects yields the maximum return while not requiring more money than is available.

At first sight, the foregoing problems might appear to possess little in common for they differ both in terms of subject matter (agriculture; industry; economic development planning) and scale (individual farmer; firm; government). Yet there are common features. Each involves finding a solution - called the <u>optimal solution</u> - which fulfils an aim or <u>objective.</u> The latter is expressed in terms of making some quantity,e.g. profit, as large - <u>maximisation</u> - or as small - <u>minimisation</u> - as possible. In each case, <u>constraints,</u> e.g. resource availabilities, limit the extent to which the objective may be fulfilled. We seek to go as far as possible towards meeting the objective while keeping within the bounds of the constraints. Problems of this class are said to be of the <u>optimisation</u> or <u>normative</u> type. The number of topics which can be viewed in terms of optimisation is very great. Some examples, taken from various areas of our subject, are listed in Table 1. Pause in your reading and think of some others.

Table 1. Examples of Geographical Optimisation Problems

Subject Area	Reference	Problem (s.t. : subject to)
Physical Geography/ Geology	I Taha (1976), 24	To determine a production schedule for N oil fields over T years to maximise profit s.t. demand for oil in each year met; no field over-produces.
Water Quality	Taha (1976), 21	To determine the necessary efficiency of each of three water treatment plants along a river to minimise construction costs s.t. maximum pollution levels not exceeded.
Irrigation	Soltani-Mohammadi (1972)	For a particular type of irrigation, to determine the amounts of various crops to grow, to maximise profit s.t. resources (land, water, labour) not over-used; market and crop rotation restrictions.
Animal Husbandry	Connolly (1974)	To determine the number of cattle and sheep to graze on a pasture to maximise profit s.t. no overgrazing of any pasture component.
Crop Choice	Henderson (1959)	To determine the amounts of various crops to grow to maximise expected return s.t. no more than acreage available used; acreage limits for specific crops not exceeded; crop pattern recommended differs from that of previous year by no more than a given amount.
Human and Animal Diets	Gould *et al.* (1969) Taylor (1977), 312	To determine the amounts of various foods to include in a diet such that diet cost is minimised s.t. minimum nutritional requirements fulfilled.
Economic Development	Isard (1960)	To determine the mix of economic activities to carry on in a region to maximise profit s.t. regional resources not exceeded.
Urban Development	Herbert *et al.* (1960)	To determine the land acreage to be allocated to various house-types in a development scheme to maximise the total savings of those to be located s.t. land availability and housing demand requirements.

Table 1. continued...

Transport	Casetti (1966)	To determine for the Southern Ontario steel market the flow patterns which minimise total transportation costs s.t. market demands for steel and resultant demands for ore and coal met; specified ore volume shipped to U.S. for internal consumption; transport resources conserved at each location.
Game Theory	Gould (1963)	To maximise expected gain or minimise expected loss in a gaming situation.

Given an apparently large number of different types of optimisation problem in terms of subject matter, you might suspect that it has proved necessary to devise many different mathematical techniques for determining optimal solutions. Yet this is not the case. Research workers, mainly in the field of <u>Operations Research</u> or <u>Management Science</u>, have shown that when an optimisation problem is expressed mathematically, it usually falls into one of a relatively small number of categories and is thus amenable to solution using one of a relatively small number of methods. The most common type of optimisation problem involves what is known as <u>linear programming</u>; the mathematical method most commonly employed to find the optimal solution to such problems is the <u>simplex method</u>.

This monograph concerns linear programming and the simplex method. First, the mathematical structure and the underlying assumptions of linear programming are defined (Section I(iii)) and a method for solving small problems, e.g. the farmer-crop problem mentioned at the outset, is described (II(i)). The simplex method is then discussed (II(ii)-(iv)), once again using the farmer-crop problem as an example. Once the optimal solution to a particular optimisation problem has been found, interest often focuses on the extent to which this solution changes for given changes in the objective and/or constraints. Such an investigation is called a <u>sensitivity analysis</u>. The various types of sensitivity analysis and their relevance are discussed in Section III. To conclude, certain special topics are considered (IV and VI) and further details of some geographical linear programming problems are presented (V).

Before proceeding further, the question should perhaps be asked: why are the notions underlying optimisation in general and linear programming in particular of interest to the geographer in the first place? The most important reason is that in many instances interesting and useful insights into geographical problems can be obtained by viewing these in terms of optimisation; in addition, recommendations concerning how current patterns and practices should be changed can be made. For the farmer-crop problem, if the amounts of two crops that should be produced in order to maximise profit can be calculated, these can then be compared with the amounts actually being produced at present in order to assess the efficiency of the farmer's current practice. In addition, recommendations can be made concerning how production should be planned in future in order to raise profit. This double role of first providing information against which current patterns and activities may be judged and second of providing a means by which reasoned proposals for the future may be made represents the main uses of optimisation in geography.

In recent years, geographers have become increasingly interested in expressing and solving problems in terms of optimisation. To some extent this reflects certain changes of emphasis within our subject. Geographers are becoming more involved in such matters as policy definition, assessing the possible implications of pursuing different policies, and in environmental management in its widest sense. An increasing emphasis is also being laid on the need to determine how resources may be used as efficiently as possible. Another topic in which there is now increased interest is that of how decisions concerning geographical phenomena, e.g. the location of a factory, are and should be arrived at. The concepts underlying optimisation appear singularly appropriate to studies in all these areas for they involve first the definition of an objective (policy) and constraints and then the determination of the optimal (most efficient) solution on the basis of which reasoned recommendations may be made and decisions taken.

Within the general subject area of optimisation, why is linear programming and the simplex method of particular interest? The main reason is that the range of actual and potential applications of these is particularly large. The problems listed in Table 1 appear dissimilar; yet, when expressed in mathematical form, they all turn out to be linear programming problems - a point that you will be able to check for yourself after you have mastered the material to follow. A further point which increases the usefulness of a study of linear programming and the simplex method is that many optimisation problems which do not fall into the linear programming category may still be solved using an amended version of the simplex method. For this reason, mastery of this method provides the necessary base from which you can proceed to study more advanced topics should you wish.

(ii) Prerequisites

It is desirable that the reader be familiar with the earlier monograph in this series (Hay, 1977) which deals with one type of linear programming problem - the transportation problem. It will be shown here that this is a linear programming problem with a special mathematical structure which enables the methods given in Hay to be employed in its solution. This monograph builds on the material presented by Hay by generalising it to cover any linear programming problem.

Mathematically, it is assumed that the reader understands the notation for summations (Σ) (Section VI (ii) only) and for inequalities ($\geq$; $\leq$) and is familiar with the idea of representing lines and planes by equations and of solving sets of equations for unknowns. Certain aspects of the latter topic are treated in the appendix. The reader is urged to perform the calculations described in the text for himself as he proceeds, thereby allowing thorough understanding.

(iii) Mathematical Structure of Linear Programming Problems

Reconsider the farmer-crop problem mentioned in Section I(i) and suppose that the actual data are as given in Table 2. We seek the quantities (in tons)

of crops 1 and 2 that should be grown in order to maximise profit while not overusing any resource. Let X_1 and X_2 designate these quantities. X_1 and X_2 are referred to as the decision variables because the purpose in solving the problem is to decide what values these variables should take.

Table 2. Farmer-Crop Problem : Input Data

Resource	Units of resource used up per ton grown of: Crop 1	Crop 2	Total Availability of Resource
Land	2	1	8
Labour	1	1	5
Water	1	2	8
Profit per ton grown	2	3	

Consider the objective of profit maximisation. If X_1 tons of crop 1 are grown, the profit accruing from crop 1 will be $2X_1$. Similarly if X_2 tons of crop 2 are grown, a profit of $3X_2$ will result. Thus the total profit gained by growing X_1 tons of crop 1 together with X_2 tons of crop 2 will be $2X_1 + 3X_2$.

Given that profit must be maximised, the objective may be written mathematically as:

Maximise : $2X_1 + 3X_2$

Consider now the first constraint, that no more land than is available may be used. If X_1 tons of crop 1 are grown, the total amount of land used for crop 1 (measured in the units on which the data in Table 2 are based) is $2X_1$. Similarly, if X_2 tons of crop 2 are grown, the total amount of land used for crop 2 is $1X_2$. Thus the total amount of land used by growing X_1 tons of crop 1 together with X_2 tons of crop 2 is $2X_1 + X_2$ which must not exceed 8 units. Thus mathematically, X_1 and X_2 must be such that:

$$2X_1 + X_2 \leq 8$$

Similarly, consideration of the constraints on labour and water yields the mathematical conditions:

$$X_1 + X_2 \leq 5$$

and

$$X_1 + 2X_2 \leq 8.$$

Finally, there is the obvious stipulation that negative amounts of a crop cannot be produced i.e.

$$X_1 \geq 0$$

$$X_2 \geq 0$$

The preceding six mathematical expressions together completely describe the optimisation problem to be solved. Two characteristics of the expressions identify it as a linear programming problem. First, the objective and each of the constraints are linear i.e. contain terms involving X_1 or X_2 alone rather than (say) X_1X_2, X_1/X_2 or X_1^2. (In the next section, it is demonstrated that the objective and constraints appear as straight lines when plotted graphically). The most important implication of this is that linear programming cannot be used unless it can be assumed: first, that resources once allocated to one activity, such as growing crop 1, cannot have any bearing on another activity, e.g. growing crop 2, (such an interdependency would produce terms such as X_1X_2 or X_1/X_2 in the expressions) and, second, that no <u>economies of scale</u> are possible, so that to (say) exactly double production of crop 1 requires exactly double the amount of each resource rather than, as might be expected with increased scale of production, somewhat less than this. To relax this characteristic and allow such nonlinear terms as X_1^2 in the formulation involves entering the field of <u>nonlinear programming</u> which is computationally more complex. The second characteristic of the preceding expressions which identify the farmer-crop problem as a linear programming problem is that within the limits of the constraints, no additional restrictions are placed on the values that X_1 and X_2 may take. Suppose in a problem that X_1 and X_2 are the number of new schools and factories respectively to be constructed. Here, in order for the final solution to be implementable, the additional condition that X_1 and X_2 be integers must be imposed. This converts our linear programming problem to an <u>integer linear programming</u> problem which, again, is harder to solve. When formulating an actual linear programming problem, it is important to satisfy oneself whether the two foregoing characteristics apply.

When expressed mathematically, the farmer-crop problem has five constraints, the final two of which are the so-called <u>non-negativity conditions.</u> Because non-negativity of the variables is usually required in optimisation problems, the corresponding non-negativity constraints are often taken as read'. We thus state (loosely) that the farmer-crop problem involves maximisation subject to three major constraints. While, as in this case, all of the major constraints in a maximisation problem are usually of the 'less than or equals' type, some 'greater than or equals' (≥) and/or 'equality' ( ) conditions can be encountered also. Similarly, with minimisation, while all of the constraints are generally of the '≥' type, some '≤' and ' ' conditions can also be encountered.

<u>II SOLVING LINEAR PROGRAMMING PROBLEMS</u>

(i) <u>Graphical Solution of the Farmer-Crop Problem</u>

We begin by considering a graphical method by which the farmer-crop problem may be solved. The decision variable values to be determined are the optimal amounts of crops 1 and 2 (X_1 and X_2 tons) to be grown. These variables are placed on the x- and y-axes respectively of a graph (Figure 1(a)). The non-negativity constraints together require that X_1 and X_2 be positive; thus the negative valued sections of the axes are omitted from the graph.

Consider now the first of the major constraints (land availability). If X_1 and X_2 are such that all available land is used up, then, mathematically:

$$2X_1 + X_2 = 8$$

This is the equation for line AA' in Figure 1(a). Geometrically, any combination of X_1, X_2 which requires exactly 8 units of land will plot out on this line; a combination requiring less than eight units will plot below it and one requiring more than eight units will plot above it (check this).

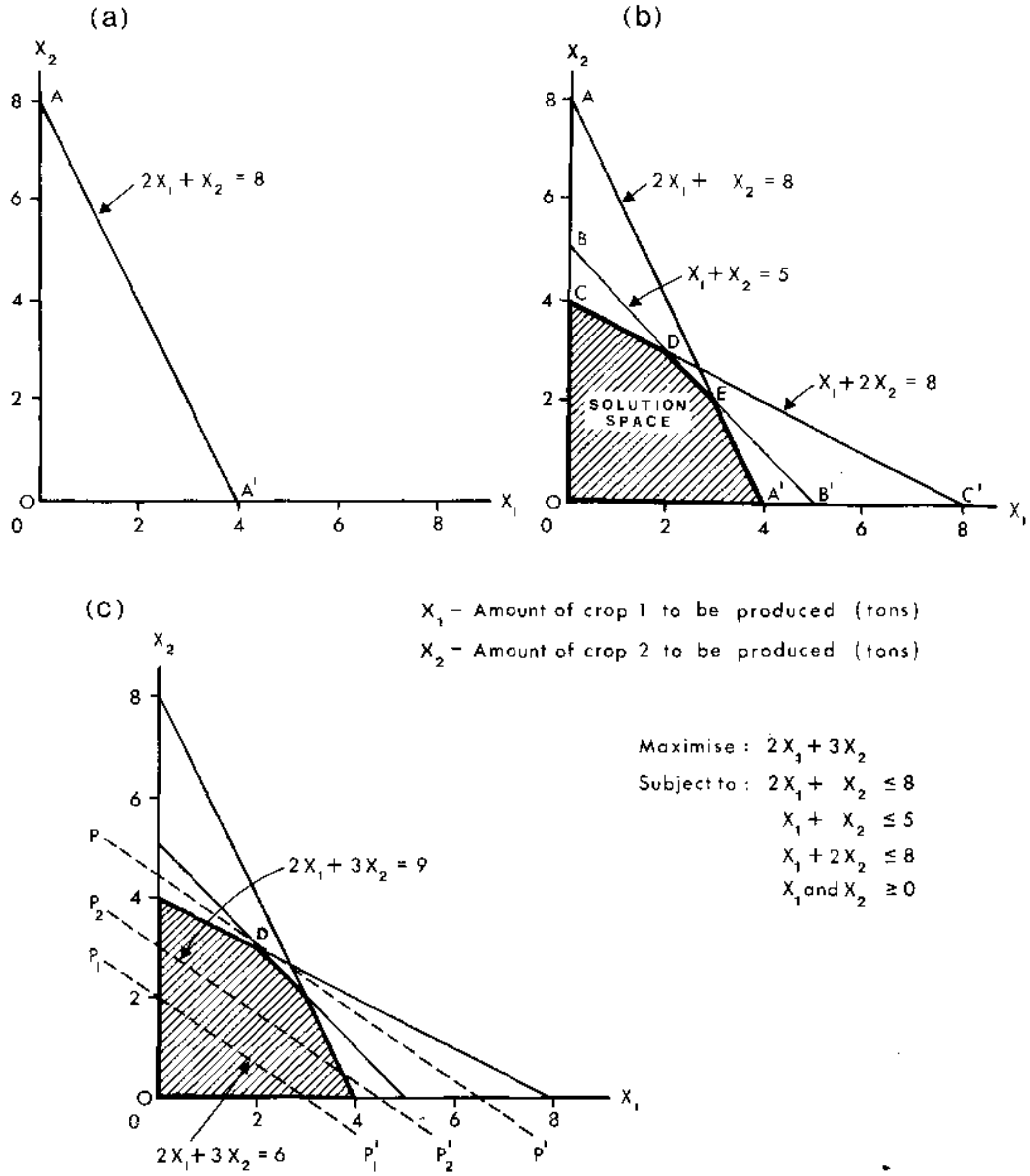

Fig. 1 Farmer-Crop Problem: Graphical Representation

To require algebraically that the optimal values of X_1, X_2 fulfil the condition:

$$2X_1 + X_2 \leq 8$$

is equivalent to stating geometrically that the optimal solution, when plotted on the graph, lies on or below AA'. Similar treatment of the other two constraints (labour and water) yields the lines:

$$X_1 + X_2 = 5$$

$$X_1 + 2X_2 = 8$$

(BB' and CC' respectively in Figure 1(b)) on or below which the optimal solution must also lie. To require that all the constraints (including X_1, $X_2 \geq 0$) hold simultaneously is equivalent to requiring that the optimal solution lies below AA', BB', CC' with X_1 and X_2 non-negative, i.e. within or on the edge of the geometrical figure OCDEA'. This area, which comprises the set of all possible solutions, is called the <u>solution space</u>.

We now seek to determine where in the solution space the objective is maximised. To begin, suppose that the farmer does not wish to maximise his profit at all but instead, is satisfied to make 6 units of money. Obviously, if this occurs, X_1 and X_2 must be such that:

$$2X_1 + 3X_2 = 6$$

which is the line P_1P_1' in Figure 1(c). Geometrically, X_1 and X_2 must be chosen so as to fall on this line if a profit of 6 units is required; they must also fall in the solution space. Suppose now that the farmer wishes to make a profit of 9 units. The set of values for X_1 and X_2 which permit this are those lying on the line (P_2P_2'):

$$2X_1 + 3X_2 = 9$$

and within the solution space. Comparison of P_2P_2' with P_1P_1' demonstrates that for an increased profit (9 rather than 6 units), the range of possible solutions which returns the profit lies on a line parallel to the initial profit line, P_1P_1', but at an increased distance away from the origin. This suggests that the optimal (maximum profit) solution will be on the line parallel to P_1P_1' which is as far from the origin as possible while still touching the solution space at at least one point. In the case of the farmer-crop example, this in the line PP' which touches the solution space at the single point D. The coordinates of this point (i.e. the optimal values for X_1, X_2) can be read directly from Figure 1(c) and are:

$$X_1 = 2 \; ; \; X_2 = 3$$

The corresponding maximum profit (let this be called X_o) may be obtained by substituting the optimal value for X_1 and X_2 in the original objective which yields:

$$X_o = 2(2) + 3(3) = 13$$

Note that in this example, the optimal solution occurs at the intersection of the constraints relating to labour and water availability. Thus it is these two resources that are being used up fully in achieving maximum profit.

Consideration of the geometry underlying a two variable linear programming maximisation problem such as the farmer-crop example suggests the important conclusion that, in general, the optimal solution to such a problem lies at one of the outermost points of the solution space i.e. at the intersection of two of the constraints. This is because when seeking the optimal solution, the profit line is moved as far as possible from the origin without leaving the solution space; at its maximum distance from the origin, the profit line generally touches the solution space at one of its outermost points e.g. C, D, E or A' in Figure 1(b). (A situation in which the optimal solution does not lie at a single outer point of the solution space is discussed in Section IV(iv) .)

The graphical method represents a straightforward way of solving linear programming problems. Its obvious drawback is that it can only be used successfully on problems concerning two decision variables; for three variables, a three-dimensional diagram with the constraints and profit lines appearing as planes would be required, which would be difficult (although not impossible) to draw; for larger numbers of unknowns, graphical representation would be impossible. The range of real-world problems to which the graphical approach can be applied is thus limited. Nevertheless, it is extremely helpful for, as will be demonstrated, it provides a framework within which the algebraic methods used to solve large linear programming problems can be understood clearly.

(ii) <u>Preliminaries to the Simplex Method</u>

A linear programming problem comprises an objective plus constraints, some or (often) all of which are inequalities. Given that most algebraic theory concerns equations (i.e. equalities) rather than inequalities, it is not surprising that the algebraic methods that have been evolved for solving linear programming problems begin by converting the constraint inequalities into equations.

Consider the constraint:

$$2X_1 + X_2 \leq 8$$

To convert this to an equality, a quantity (let this be called S_1) must be added to the left hand side to raise it to the level of the right giving:

$$2X_1 + X_2 + S_1 = 8$$

Note that in creating the equality, a new variable, S_1, has had to enter the system. Consider now the inequality:

$$X_1 + 3X_2 \geq 3$$

Here, to convert to an equality, a quantity (let this be called S_2) must be subtracted from the left hand side to reduce it to the level of the right giving:

$$X_1 + 3X_2 - S_2 = 3$$

In both of the preceding examples, S_1 and S_2 are referred to as <u>slack variables</u> for they counteract the difference or slack between the left and right hand sides of the inequalities.

Consider now the equations:

$$\left.\begin{aligned} 2X_1 + 3X_2 &= 8 \\ 3X_1 + 5X_2 &= 13 \end{aligned}\right\} \text{--------(1)}$$

Because there are two equations involving two unknowns (X_1, X_2), they can be solved to yield unique values for X_1 and X_2 (1 and 2) which satisfy both equalities simultaneously (see Appendix). Consider now the pair of equations:

$$\left.\begin{aligned} 2X_1 + X_2 + S_1 + 0.S_2 &= 8 \\ 3X_1 + 3X_2 + 0.S_1 + S_2 &= 15 \end{aligned}\right\} \text{-----(2)}$$

Here, there are two equations with four unknowns and so unique values for X_1, X_2, S_1, S_2 cannot be found. Suppose, however, that X_1 and X_2 are known to equal zero. Then, the terms containing X_1 and X_2 disappear and values for S_1 and S_2 can be found. Indeed, given the way in which the equations (2) happen to have been written (with, for S_1 a coefficient of 1 in the first equation and zero in the second and, for S_2, vice versa), the values of S_1 (=8) and S_2 (=5) can be read directly.

Suppose now that instead of X_1 and X_2 equalling zero, it is known that in (2) X_2 and S_2 are zero, i.e. S_2 has changed roles with X_1. In order to be able to read off the values of X_1 and S_1 as conveniently as before, the equations (2) would have to be adjusted to appear in the form:

$$\left.\begin{aligned} 0\,X_1 + \alpha_{11}X_2 + 1\,S_1 + \alpha_{12}S_2 &= \beta_1 \\ 1\,X_1 + \alpha_{21}X_2 + 0\,S_1 + \alpha_{22}S_2 &= \beta_2 \end{aligned}\right\} \text{-----(3)}$$

where α_{11}, α_{12}, α_{21}, α_{22}, β_1, β_2 represent the numerical values that these coefficients must have in order that the equation pairs (2) and (3) be equally valid. Conversion of (2) to the format of (3) may be achieved using successively two mathematical operations which may be applied to equation sets such as these, namely:

(i) A particular equation can be multiplied or divided across by a chosen constant

(ii) A particular equation can be multiplied or divided across by a chosen constant and then added to or subtracted from another equation. A detailed discussion of these rules is given in the Appendix.

In order to convert (2) to the format of (3) using (i) and (ii), first identify the equation in (2) in which a particular coefficient is to take a value of one whereas another, currently of value one, may alter. In this case, the equation is that in the second row where it is required that the coefficient of X_1 become one whereas that of S_2, currently equal to one, may alter. In order to create a coefficient of one for X_1, divide the second equation in (2) across by three which yields:

$$\left.\begin{aligned} 2X_1 + X_2 + S_1 \quad &= 8 \\ X_1 + X_2 + 1/3S_2 &= 5 \end{aligned}\right\} \text{--------(4)}$$

Next, in order to create a coefficient of zero for X_1 in the first row of (4) multiply the second equation in (4) across by two and subtract it from the first which yields:

$$\left.\begin{aligned} - X_2 + S_1 - 2/3S_2 &= -2 \\ X_1 + X_2 + 1/3S_2 &= 5 \end{aligned}\right\} \text{--------(5)}$$

The equation set (5) is now in the format required by (3). With $X_2 = S_2 = 0$, it can be seen immediately that $X_1 = 5$ and $S_1 = -2$.

For a set of equations, where some variables are known to be zero, the non-zero variables are said to form the basis of the equations. Thus in (2), S_1 and S_2 form the basis. The exchange of one variable for another in the basis, e.g. S_2 for X_1 in the example, is called a change of basis operation. Mathematically, this operation involves first creating a coefficient of one for the variable entering the basis in the row of the variable leaving it (e.g. for X_1 in the (second) row in which S_2 currently has a coefficient of one) and then of creating zero coefficients for the entering variable (e.g. for X_1) in the other row(s). By definition, the equations are said to have been pivoted on the term where a coefficient of one was created initially as on the term $3X_1$ in the example. It will be shown that change of basis operations play a key role in the solving of linear programming problems.

When dealing with change of basis operations, it has been found convenient to summarise the sets of equations concerned in tabular form. For the initial equations in the preceding example (2), this gives:

BASIS	X_1	X_2	S_1	S_2	SOLUTION
S_1	2	1	(1)	0	8
S_2	[3]	3	0	(1)	15

The coefficients of the variables are placed below the variables concerned, the right hand sides of the equations being referred to as the solution. The variables forming the basis are listed on the left. By convention, the coefficients of one via which values for the basis variables may be read off are circled to highlight them. The variables in the initial basis, S_1 and S_2, are separated by vertical lines from the non-basis variables and the solution column. If it is intended that X_1 enter the basis and S_2 leave, a square is placed around the coefficient upon which pivoting is to occur i.e. that in the row labelled S_2 and column labelled X_1. Following the change of basis operation which yields the equations (5), the revised table is:

BASIS	X_1	X_2	S_1	S_2	SOLUTION
S_1	0	-1	(1)	-2/3	-2
X_1	(1)	1	0	1/3	5

from which, as before, $S_1 = -2$ and $X_1 = 5$ if $X_2 = S_2 = 0$.

We now consider some important relationships between linear programming constraints expressed as equations and solution space geometry. Consider again the constraints of the farmer-crop problem, which, employing slack variables, may be written in equation form as:

$$\begin{aligned} 2X_1 + X_2 + S_1 \qquad\qquad &= 8 \\ X_1 + X_2 \qquad + S_2 \qquad &= 5 \\ X_1 + 2X_2 \qquad\qquad + S_3 &= 8 \end{aligned}$$

For the first constraint and for any point on the constraint line itself:

$$2X_1 + X_2 = 8$$

and thus $S_1 = 0$. Below the constraint line:

$$2X_1 + X_2 < 8$$

and thus S_1 is positive (and increases as distance from the line increases). Similarly, above the line, S_1 is negative. In the same way, S_2 and S_3 are zero for points on the constraint lines to which they relate, positive below them and negative above them. Consideration of all the constraints at once (including the non-negativity constraints) leads to the conclusion that the solution space (Figure 1(b)) comprises the set of points for which X_1, X_2, S , S_2, S_3 are all greater than or equal to zero for, by definition, passing out of the solution space involves going above one or more of the constraints (S_1 and/or S_2 and/or S_3 become negative) and/or into the region where X and/or X_2 are negative. For a point actually on one of the boundaries of the solution space, one of X_1, X_2, S_1, S_2, S_3 must equal zero for, by definition, such a point lies either on a graph axis (X_1 or $X_2 = 0$) or on a constraint line (S_1 or S_2 or $S_3 = 0$). Finally, for the outermost points of the solution space i.e. O, C, D, E, A' in Figure 1(b), two of X_1, X_2, S_1, S_2, S_3 must equal zero for, by definition, these points are formed by the intersection of two constraints/axes. This conclusion is of particular importance for, as demonstrated previously, the optimal solution to a linear programming problem generally occurs at one of these points.

The farmer-crop problem concerns two decision variables (X_1, X_2) with three constraints which introduce three slack variables (S_1, S_2, S_3), giving a total of five. In the optimal solution it is anticipated that two of the five variables will equal zero with the other three being positive. In general, for a linear programming problem comprising n decision variables with m constraints which introduce m additional slack variables to yield a total of (m + n), n of the (m + n) variables can be expected to equal zero with the remaining m being positive in the optimal solution.

(iii) Solution of the Farmer-Crop Problem Using the Simplex Method

The ideas presented in the previous section are now combined to provide a method (called the simplex method) for solving the farmer-crop problem and then, more importantly, any linear programming problem. To begin, reconsider the farmer-crop problem objective:

$$\text{Maximise : } 2X_1 + 3X_2 \ (= X_0)$$

together with the mathematical statement obtained by subtracting this objective from X_0 and setting the result to zero i.e.

$$X_0 - 2X_1 - 3X_2 = 0 \quad \text{----------(6)}$$

Suppose $X_1 = 2$ and $X_2 = 3$. (This is in fact the optimal solution to the farmer-crop problem.) Then, for the equality (6) to hold, $X_0 = 2(2) + 3(3) = 13$ which is in fact the optimal value of the objective. Again, if $X_1 = X_2 = 1$ which is a feasible but non-optimal solution to the farmer-crop problem, $X_0 = 5$ which again is the objective value for these values of X_1 and X_2. In general, for any values of X_1 and X_2, the above equality is such that X_0 automatically gives the corresponding value of the objective for those values of X_1 and X_2. In this sense, it represents the objective in equation form.

Given methods for converting both the objective and the constraints to equalities, the farmer-crop problem may be rewritten:

$$\text{Solve: } \left.\begin{array}{l} X_0 - 2X_1 - 3X_2 = 0 \\ 2X_1 + X_2 + S_1 = 8 \\ X_1 + X_2 + S_2 = 5 \\ X_1 + 2X_2 + S_3 = 8 \end{array}\right\} \text{----- (7)}$$

It is known that in the optimal solution, two of X_1, X_2, S_1 S_2, S_3 will equal zero with the other three variables exceeding zero and that the appropriate value of X_0 will be generated automatically via the first equation in (7).

The four preceding equations may be viewed conveniently in terms of the ideas presented in the previous section. Given their format, with X_0, S_1, S_2 and S_3 each having a coefficient of one in one equation and zero in the others, these variables may be viewed as a basis yielding the solution:

$X_0 = 0$; $S_1 = 8$; $S_2 = 5$; $S_3 = 8$ (Basis variables)

$X_1 = 0$; $X_2 = 0$ (Non-basis variables)

Because two of X_1, X_2, S_1, S_2, S_3 equal zero, it is known that this is a possible solution to the linear programming problem i.e. it corresponds to an outer point of the solution space. Geometrically, with $X_1 = X_2 = 0$, this point is the origin (Figure 1(b)).

Before proceeding, and following the principles presented in the previous section, the equations (7) may be summarised in tabular form :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	1	-2	-3	0	0	0	0	
S_1	0	2	1	1	0	0	8	8
S_2	0	1	1	0	1	0	5	5
S_3	0	1	2	0	0	1	8	4

Note that by convention, the row and column relating to the objective (X_0) are separated by a vertical and horizontal line respectively. Ignore the 'ratio' column for the moment. In the context of linear programming, a tabular representation of a problem in this format is called a simplex tableau.

In order to discover a solution to the farmer-crop problem which yields a higher profit (i.e. X_0 value) than $X_1 = X_2 = 0$, we view the constraint equations in the simplex tableau in terms of a change of basis operation. Such an operation would involve one of X_1, X_2 (being the variables currently outside the basis) entering it and one of S_1, S_2, S_3 leaving. Consider first the choice of an entering variable. To allow X_1 or X_2 to enter the basis and take a positive value is equivalent in real terms to permitting the farmer to commence growing either crop 1 or crop 2. From the viewpoint of the farmer desiring to maximise profit, crop 2 is the better choice for, as may be seen from the objective function or from the first row of the simplex tableau (recalling that a positive value in the original objective takes a negative sign when converted to equation form), each ton of crop 2 produced yields a profit of 3 units as against only 2 units for crop 1. This suggests the following general rule for choosing a variable to enter the basis in a change of basis operation on a simplex tableau:

<u>RULE 1 :</u> In a maximisation problem, choose as the variable to enter the basis that with the largest negative coefficient in the objective row (in the case of a tie, choose either variable).

Consider now the choice of a variable to leave the basis. Given that X_2 is to enter the basis and still bearing in mind that profit is to be maximised, the farmer would wish to grow as much of crop 2 as possible (i.e. make X_2 as large as possible) within the bounds of the resources available (i.e. constraints). Inspection of the geometry of the solution space (Figure 1(b)) demonstrates that in terms of the land constraint X_2 may increase from 0 to 8, in terms of the labour constraint from 0 to 5, and in terms of the water constraint from 0 to 4. Consideration of the three limits simultaneously shows that the severest restriction (as given by the minimum of the three upper limits) is imposed by the constraint relating to water : when X_2 reaches 4 i.e. the point C of the solution space, the solution lies on this constraint and, algebraically, $S_3 = 0$. This demonstrates that S_3 is the variable to leave the basis.

In terms of the simplex tableau, the testing of each constraint followed by identification of the variable to leave the basis may be conducted as follows:

<u>RULE 2 :</u> For each constraint in turn, divide the entry in the 'solution' column by the entry in the column of the variable entering the basis, recording the result in the 'ratio' column. <u>Ignore all negative ratios.</u> The row variable corresponding to the smallest positive ratio is the one to leave the basis. (In the case of a tie choose either variable.) Note that the ratios obtained by applying this rule to the farmer-crop problem correspond to the amounts by which X_2 can increase within the bounds of each constraint and thus that application of rule two is the algebraic equivalent of the geometrical steps discussed above.

Given that X_2 is to enter the basis and that S_3 is to leave and following the methods of the previous section, it is required that the equations be pivoted on the term in the row labelled S_3 and column labelled X_2. This is achieved by dividing the S_3 row throughout by two (to yield the required coefficient of one) and then by multiplying the revised row in turn by -3, 1 and 1 and subtracting from, in turn, the X_0, S_1 and S_2 rows to give (check this):

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	1	-1/2	0	0	0	3/2	12	
S_1	0	3/2	0	1	0	-1/2	4	8/3
S_2	0	1/2	0	0	1	-1/2	1	2
X_2	0	1/2	1	0	0	1/2	4	8

From the tableau, the new solution is:

$X_0 = 12$; $S_1 = 4$; $S_2 = 1$; $X_2 = 4$ (Basis variables)

$X_1 = 0$; $S_3 = 0$ (Non-basis variables)

which, as expected, corresponds to the point C on the graph (Figure 1(b)).

Consider now the possibility of further improving the solution (i.e. of further raising X_0) <u>via</u> another change of basis operation. Inspection reveals one negative value (for X_1) in the X_0 row of the simplex tableau which, following rule one, demonstrates that X_1 should enter the basis. The magnitude of the negative value, -1/2, indicates that for each unit increase in X_1 with adjustment to X_2 to keep within the bounds of the constraints, X_0 will increase by 1/2. The relative magnitudes of the ratios between the solution and X_1 columns indicates that S_2 should leave the basis. The pivoting operation on the term in the row labelled S_2 and column labelled X_1 involves multiplying the S_2 row across by 2 and then multiplying the revised row in turn by -1/2, 3/2, 1/2 and subtracting from the X_0, S_1, X_2 rows respectively which yields (check this):

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	1	0	0	0	1	1	13	
S_1	0	0	0	1	-3	1	1	
X_1	0	1	0	0	2	-1	2	
X_2	0	0	1	0	-1	1	3	

which corresponds to the solution:

$X_0 = 13$; $S_1 = 1$; $X_1 = 2$; $X_2 = 3$ (Basis variables)

$S_2 = 0$; $S_3 = 0$ (Non-basis variables)

This solution corresponds to the point D in Figure 1 and is in fact the optimal solution. The latter point can be verified from the revised simplex tableau by noting that no variable in the X_0 row now has a negative coefficient and could thus enter the basis and raise X_0. For completeness, rule one must be extended to read in full:

<u>RULE 1 :</u> In a maximisation problem, choose as the variable to enter the basis that with the largest negative coefficient in the objective row; (in the case of a tie, choose either variable). If there are no negative coefficients in the X_0 row, the optimum has been defined.

The complete simplex procedure for a maximisation problem with all 'less than or equals' constraints is summarised in Figure 2. Algebraically, the method may be viewed as involving a series of change of basis operations performed on simplex tableaus. On each occasion, the entering variable is chosen as that offering the greatest per unit increase in the objective. Geometrically, the simplex method may be viewed as commencing at the origin and of proceeding logically along the outer edges of the solution space from outer point to outer point until the optimal solution location is reached.

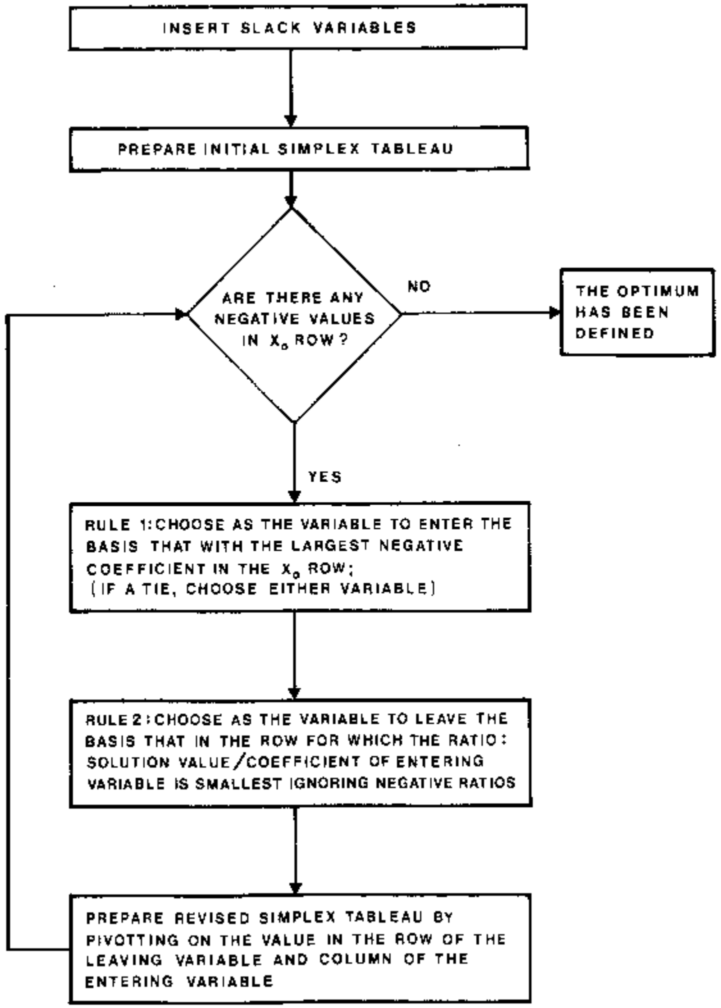

Fig. 2 Summary of Simplex Procedure for a Maximisation Problem with all 'less than or equals' constraints

The simplex method has been used here to solve a linear programming problem comprising two decision variables and three constraints. The methods discussed can be used to solve a linear programming maximisation problem comprising (within reason) any number of decision variables and less than or equals constraints. The only difference would be the increased size of the simplex tableaus. This ability to deal with problems of any size represents the most important advantage of the simplex method over the graphical approach.

Two further features of the method should be noted. First, the optimal solution is approached via a number of steps or iterations, each bringing one further (by means of an increase in X_0) towards the optimal solution. Many optimisation methods proceed in this manner. Second, given that the method involves finding the optimal solution by 'tracking' successively from one outer point of the solution space to another, the rate at which the solution is likely to be reached in a particular problem obviously depends very much on the number of such points which, given that these occur at the intersections of constraint lines, is related to the number of constraints. Thus, in general, a linear programming problem with many constraints and few variables is more time consuming to solve using the simplex method than one with many variables and few constraints.

(iv) Generalisation of the Simplex Method

In the preceding section, the simplex method is applied to a linear programming maximisation problem with two (and ultimately any number of) decision variables and with three (and ultimately any number of) constraints. We now consider how the method may be adapted to deal with the other constraint types (>= and =) and with minimisation thereby providing a technique for solving any linear programming problem. The only case requiring additional theory concerns '>=' constraints.

Consider the linear programming problem:

Maximise : $2X_1 + 3X_2$

Subject to (s.t.) : $X_1 + 3X_2 \geq 3$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$ $\qquad X_1, X_2 \geq 0.$

This is the farmer-crop problem but with the first constraint replaced by a relationship of the '≥' type. In reality, such a constraint could represent (say) a stipulation by government that at least a certain amount of crop 1 and/or 2 be grown. Graphical representation of the problem (Figure 3) demonstrates that the optimal solution is the same as before. The major difference is that the new constraint excludes an area including the origin from the solution space (ACDB'A'). Quite obviously, this has implications for the simplex method for, as demonstrated, the initial simplex tableau has X_1 and X_2 as non-basis variables (i.e. equal to zero), which corresponds geometrically to the solution being at the origin.

Algebraically, the difficulty raised by the 'greater than or equals' constraint can be seen by rewriting the problem in equation form, which yields:

Solve:

$$X_0 - 2X_1 - 3X_2 = 0$$
$$X_1 + 3X_2 - S_1 = 3$$
$$X_1 + X_2 + S_2 = 5$$
$$X_1 + 2X_2 + S_3 = 8$$

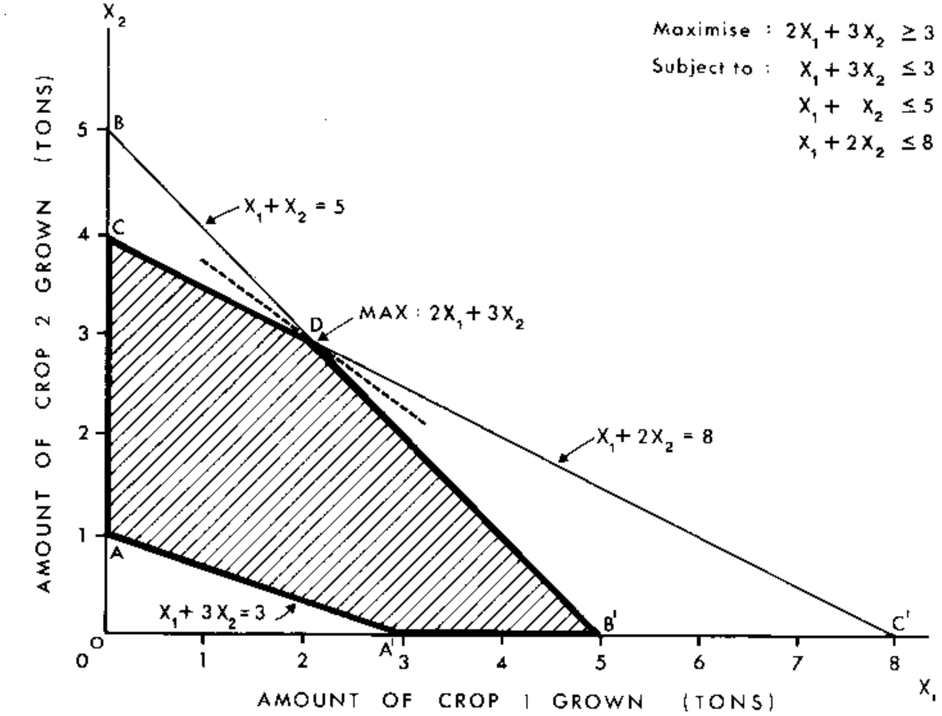

Fig. 3 Graphical representation of problem with one 'greater than or equals' constraint

As before, considering X_0, S_1, S_2, S_3 as the basis yields the solution:

$X_0 = 0$; $S_1 = -3$; $S_2 = 5$; $S_3 = 8$ (Basis variables)

$X_1 = 0$; $X_2 = 0$ (Non-basis variables)

which is not permissible because one of the constraint variables, S_1, is negative which indicates that the point to which it refers, 0, is not in the solution space.

In order to surmount this difficulty and, in particular, to produce an allowable initial solution with X_1 and X_2 outside the basis (i.e. equal to zero as required) consider the problem:

Maximise: $2X_1 + 3X_2 - MR_1$

s.t.: $X_1 + 3X_2 + R_1 \geq 3$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$ $X_1, X_2, R_1 \geq 0$

where R_1 is a new variable and M is a very large number (say one million).

This is in fact the previous problem with one times R_1 added to the greater than or equals constraint and -M times R_1 added to the objective. What value would R_1 be expected to take in the optimal solution ? Given that the coefficient of R_1 in the objective is -M and that this is a maximisation problem, even a small value of R_1 would dramatically reduce the objective value. Thus, unless there are exceptional circumstances, R_1 will equal zero in the optimal solution. If this is so, the problem remaining (and thus its solution) is effectively the initial problem. In a sense, the variable R_1 appears superfluous for it has been formulated in such a way as to ensure its disappearance!

The major use of R_1 is that it offers a means of providing an initial simplex tableau. Expressing the revised problem in tableau form with R_1 replacing S_1 yields:

X_0	X_1	X_2	S_1	R_1	S_2	S_3	SOLUTION
1	-2	-3	0	M	0	0	0
0	1	3	-1	1	0	0	3
0	1	1	0	0	1	0	5
0	1	2	0	0	0	1	8

With one exception, the M in the R_1 column of the objective row, this tableau may be viewed as a solution with R_1, S_2 and S_3 as basis. The M term may be removed by multiplying the R_1 row by M and subtracting the product from the X_0 row to give:

BASIS	X_0	X_1	X_2	S_1	R_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	-2-M	-3-3M	M	0	0	0	-3M	
R_1	0	1	3	-1	(1)	0	0	3	1
S_2	0	1	1	0	0	(1)	0	5	5
S_3	0	1	2	0	0	0	(1)	8	4

and thus the solution:

$X_0 = -3M$; $R_1 = 3$; $S_2 = 5$; $S_3 = 8$ (Basis variables)

$X_1 = 0$; $X_2 = 0$; $S_1 = 0$ (Non-basis variables)

which is permissible because all the constraint variables (decisions and slacks) are greater than or equal to zero.

In order to investigate why the introduction of R_1 permits a feasible solution with $X_1 = X_2 = 0$, consider the geometry of the extended problem. Because there are three decision variables (X_1, X_2, R_1), a three dimensional representation is required (Figure 4). The constraint:

$X_1 + 3X_2 + R_1 \geq 3$

requires that the optimal solution lies above the sloping plane AA'A" whereas the less than or equals constraints together require that it lies below two

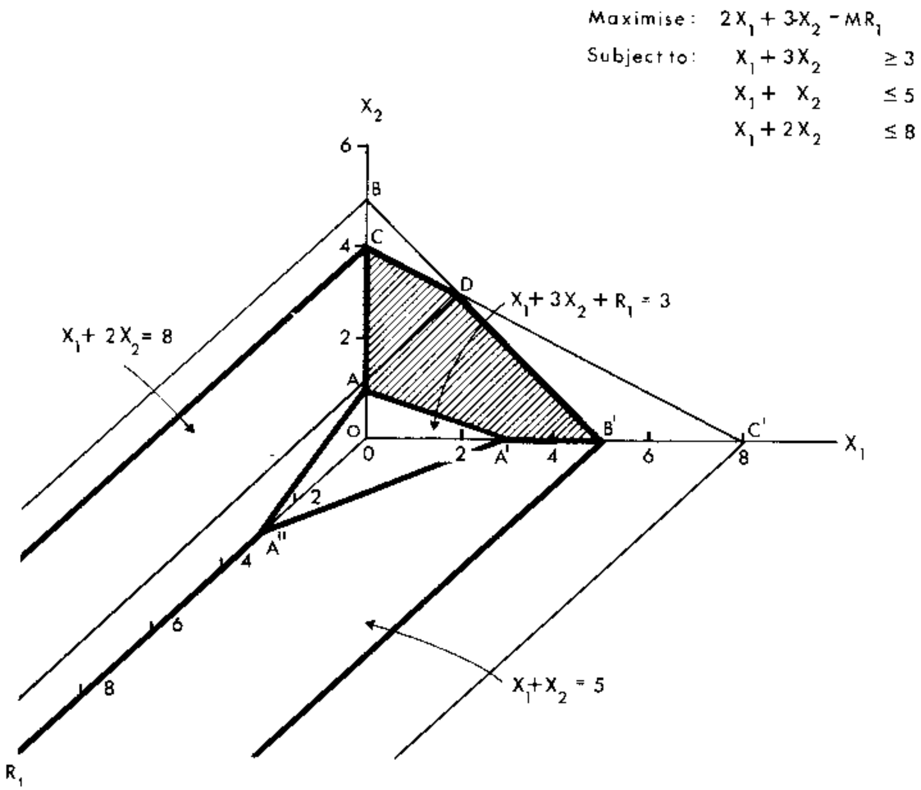

Fig. 4 Graphical representation of extended problem with one greater than or equals' constraint

planes meeting at an angle. The solution space in the plane X_1, X_2 (shaded) is as in the previous problem; it is open ended in the direction R_1. The initial solution corresponds to the point A". Thus the overall effect of introducing R_1 is to extend the solution space into a third dimension and to provide a solution space corner point with $X_1 = X_2 = 0$.

Returning to the intial tableau, inspection of the X_0 row demonstrates that X_2 should enter the basis while R_1 should leave. Pivoting yields (check this):

BASIS	X_0	X_1	X_2	S_1	R_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	-1	0	-1	1+M	0	0	3	
X_2	0	1/3	(1)	-1/3	1/3	0	0	1	-ve.
S_2	0	2/3	0	1/3	-1/3	(1)	0	4	12
S_1	0	1/3	0	2/3	-2/3	0	(1)	6	9

the solution of which corresponds to the point A in Figure 4. Notice that the solution has now entered the X_1, X_2 plane i.e. the plane of the original problem with, as expected, $R_1 = 0$.

Viewing the revised simplex tableau in terms of a further change of basis operation presents two equally suitable candidates for entry to the basis: X_1 and S_1. Arbitrarily choosing S_1, with S_3 left as it is associated with the smallest positive ratio, yields after pivoting:

BASIS	X_0	X_1	X_2	S_1	R_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	-1/2	0	0	M	0	3/2	12	
X_2	0	1/2	(1)	0	0	0	1/2	4	8
S_2	0	1/2	0	0	0	(1)	-1/2	1	2
S_1	0	1/2	0	(1)	-1	0	3/2	9	18

the solution to which corresponds to the point C in Figure 4. One further iteration in which X_1 enters the basis and S_2 leaves yields the final solution.

The preceding example demonstrates that the major effect of '≥' constraints in a linear programming problem is to force additional variables into the problem, one for each constraint, which obviously increases its size and complexity. Thus problems with many '≥' constraints tend to require a longer solution time using the simplex method.

We consider now how the simplex method may be adapted to deal with equality constraints and with a minimisation objective. The former are incorporated by converting each equality to two inequalities both of which mus hold simultaneously and on which the methods discussed previously may be use If, in a linear programming problem, it is a condition that:

$$X_1 + X_2 = 5$$

then:

$$X_1 + X_2 \leq 5$$

$$X_1 + X_2 \geq 5$$

must hold simultaneously. The latter two constraints replace the equality for entry to the initial simplex tableau.

Turning to the case of a minimisation objective, recall that for maximisation the variable chosen to enter the basis on each iteration is that offering the greatest per unit increase in X_0 i.e. that with the largest negative value in the X_0 row of the simplex tableau. For minimisation, what is obviously required is the variable offering the greatest per unit decrease in X_0 i.e. that with the largest positive value in the X_0 row of the simplex tableau. Rule one thus becomes:

RULE ONE : In a minimisation problem, choose as the variable to enter the basis that with the greatest positive coefficient in the objective row (in the case of a tie, choose either variable). If there are no positive coefficients in the X_0 row, the optimum has been defined.

A further alteration of technique imposed by a minimisation objective concerns the incorporation in the objective of extra dummy variables associated with greater than or equals constraints. Recall that for maximisation it was argued that -M times R_1 should be added to the objective to ensure that R_1 would be zero in the optimal solution; for minimisation, the same argument requires that +M times R_1 be added to the objective.

To illustrate the foregoing points, consider:

$$\text{Minimise:} \quad 8Y_1 + 5Y_2 + 8Y_3$$
$$\text{s.t.:} \quad 2Y_1 + Y_2 + Y_3 \geq 2$$
$$Y_1 + Y_2 + 2Y_3 \geq 3 \qquad Y_1, Y_2, Y_3 \geq 0$$

Two 'greater than or equals' constraints call for the introduction of two additional variables; let these be R_1 and R_2. Bearing in mind that the objective is of the minimisation type, the extended problem is:

$$\text{Minimise:} \quad 8Y_1 + 5Y_2 + 8Y_3 + MR_1 + MR_2$$
$$\text{s.t.:} \quad 2Y_1 + Y_2 + Y_3 + R_1 \geq 2$$
$$Y_1 + Y_2 + 2Y_3 + R_2 \geq 3 \qquad Y_1, Y_2, Y_3, R_1, R_2 \geq 0$$

Note that the positive sign for the coefficient M is justified by the fact that this is a minimisation problem. Unless there are exceptional circumstances, R_1 and R_2 will be zero in the optimal solution for if either R_1 or R_2 has even a small positive value, the objective value would increase considerably. Converting the extended problem to equation form and designating the slack variables associated with the constraints as S_1' and S_2' respectively yields:

$$\text{Solve:} \quad Y_0 - 8Y_1 - 5Y_2 - 8Y_3 - MR_1 - MR_2 = 0$$
$$2Y_1 + Y_2 + Y_3 - S_1' + R_1 = 2$$
$$Y_1 + Y_2 + 2Y_3 - S_2' + R_2 = 3$$

which in simplex tableau form (after adjustments to create zero coefficients for R_1 and R_2 in the Y_0 row) is (check this):

BASIS	Y_0	Y_1	Y_2	Y_3	S_1'	S_2'	R_1	R_2	SOLUTION
Y_0	(1)	-8+3M	-5+2M	-8+3M	-M	-M	0	0	5M
R_1	0	2	1	1	-1	0	(1)	0	2
R_2	0	1	1	2	0	-1	0	(1)	3

Because this is a minimisation problem, the first variable to enter the basis should be that with the largest positive coefficient; in this example, given a tie, Y_1 or Y_3 could be chosen.

III SENSITIVITY TESTING AND THE DUAL

(i) Types of Sensitivity Analysis

Up to this point, the major reason for solving a linear programming problem has been presented as finding the optimal solution. Often, however, this is merely the first step in a linear programming analysis for, as mentioned at the outset, once the optimal solution has been found, we wish to investigate, *via* what is called sensitivity analysis, how it would change for given changes in the inputs. As will be indicated, a major advantage of the simplex method is that once the final stage tableau has been determined, the various types of sensitivity analysis question can be answered without re-solving the problem.

Five different types of sensitivity analysis question which correspond to the five major inputs to any linear programming problem may be asked, namely - how would the optimal solution alter for given changes in the:

(1) Right hand sides of constraints (e.g. for a unit increase in the amount of land available in the farmer-crop problem);

(2) Objective coefficients (e.g. for a unit increase in the profit per ton grown of crop 1 in the farmer-crop problem);

(3) Technological coefficients this being the term applied to the left hand side coefficients in the constraints (e.g. if the amount of land needed to grow one ton of crop 1 decreased by one unit in the farmer-crop problem);

(4) Number of Decision Variables (e.g. if a third crop with known profit and technological coefficients was introduced into the farmer-crop problem); and

(5) Number of Constraints (e.g. if a constraint was added to or deleted from the farmer-crop problem).

The general notion underlying sensitivity analysis is particularly attractive to the geographer who so often is involved in assessing the effects of changes in one set of factors (e.g. profit levels, resource availabilities) on others (e.g. preferred landuse patterns as expressed by the relative amounts of various crops grown). Only the first type of sensitivity question, i.e. concerning right hand side coefficients, is considered in detail here. It should be remembered that the other types of sensitivity question exist and, more important, that they may all be investigated almost as easily as that concerning right hand side coefficients. (For a discussion see for example Taha (1976), Chapter 4.)

(ii) Sensitivity Analysis of Right Hand Side Constraint Coefficients

Recall the optimal solution to the farmer-crop problem (Figure 1(c)):

$$X_0 = 13 \; ; \; X_1 = 2 \; ; \; X_2 = 3$$

which lies at the intersection of the labour and water constraints. Suppose that one extra unit of water was available making a total of nine units and altering the third constraint of the farmer-crop problem to:

$$X_1 + 2X_2 \leq 9.$$

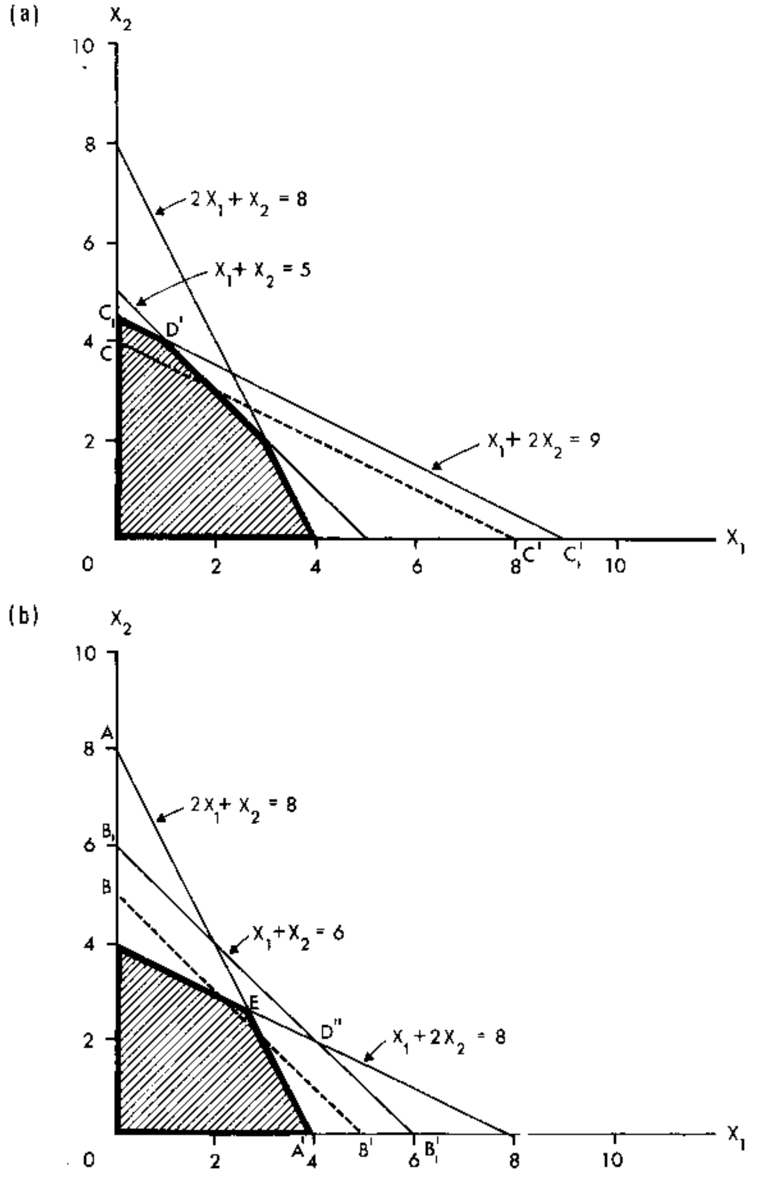

Fig. 5 Sensitivity Analysis Examples

Graphically, (Figure 5(a)) the water constraint would move to C_1C_1' and the optimal solution to D' giving:

$$X_0 = 14 \; ; \; X_1 = 1 \; ; \; X_2 = 4.$$

Thus an additional unit of water would yield a profit increase of one money unit. In this sense, it can be said that the 'economic worth' of one additional unit of water is one additional money unit.

Consider now the effect on the optimal solution of one additional land unit i.e. of altering the first constraint of the farmer crop problem to:

$$2X_1 + X_2 \leq 9.$$

Graphically (Figure 1(b)), movement of this constraint outwards from the origin has no effect on the optimal solution, for the land constraint is not one of those constraining it. Thus, it could be said that the 'economic worth' of one additional unit of land is zero money units.

Finally, consider the effect on the optimal solution of one additional unit of labour, i.e. of altering the second constraint of the farmer-crop problem to:

$$X_1 + X_2 \leq 6.$$

Graphically (Figure 5(b)), the labour constraint would move to B_1B_1' and the optimal solution, if it were still constrained by the labour and water constraints, to D" i.e.

$$X_0 = 14 \; : \; X_1 = 4 \; : \; X_2 = 2.$$

This would lead to the conclusion that the 'economic worth' of one additional unit of labour is one money unit. In fact, the optimal solution does not reach D". Instead, with the labour constraint moved outwards, the land constraint comes into play; algebraically, the basis changes and the optimal solution occurs at E.

The results of the three foregoing cases are summarised in Table 3. Notice that the information in the body of this table corresponds exactly to that in the S_1, S_2, S_3 columns of the X_0 row of the final stage simplex tableau which suggests the following important interpretation rule which can be verified mathematically:

RULE : For the final stage simplex tableau of a linear programming maximis problem, the values in the X_0 row and in the columns of the variables forming the initial basis (e.g. S_1, S_2, S_3) give the amounts by which the objective would increase for a unit increase in the right hand side of the constraint of which the initial basis variable is a part, <u>assuming that no change of basis occurs</u> during the increase.

These X_0 row values are referred to variously as <u>shadow costs</u>, <u>dual variable values</u> and <u>simplex multipliers</u>. For a minimisation problem, they give the decrease in X_0 for a unit decrease in the right hand side of the corresponding constraint. If, because of '$\geq$' constraints in the original problem, M values appear as part of the shadow costs (for an example see Section III (iii)), these should be ignored.

The most important general point to emerge from the foregoing analysis is that by virtue of its nature, the simplex method provides <u>via</u> the final tableau not only the optimal solution but also valuable information concerning

Table 3. Summary of Resource Availability Sensitivity Analyses

	Increase in X_0 for a unit increase in		
	LAND	LABOUR	WATER
Corresponding variable in initial basis	S_1	S_2	S_3
Increase	0	1	1
N.B. ASSUMING NO CHANGE OF BASIS			

the sensitivities of the right hand sides of the constraints. Investigation of the mathematics underlying the other types of sensitivity question demonstrates that they too can be answered directly or almost directly from the final stage simplex tableau. This ability to yield sensitivity information automatically represents one of the most attractive features of the simplex method.

(iii) The Dual Linear Programming Problem

Reconsider the original farmer-crop problem (Problem A) and the minimisation problem in Section II(iv) (Problem B) i.e.

PROBLEM A

Maximise : $2X_1 + 3X_2$

s.t. : $2X_1 + X_2 \leq 8$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$

$X_1, X_2 \geq 0$

PROBLEM B

Minimise : $8Y_1 + 5Y_2 + 8Y_3$

s.t. : $2Y_1 + Y_2 + Y_3 \geq 2$

$Y_1 + Y_2 + 2Y_3 \geq 3$

$Y_1, Y_2, Y_3 \geq 0$

Viewing these problems simultaneously suggests that they are in a sense 'opposites' for:

(1) Problem A concerns maximisation whereas problem B concerns minimisation;

(2) The problem A constraints are all of the 'less than or equals' type and those of problem B are all of the 'greater than or equals' type;

(3) The objective coefficients of problem A comprise the right hand side values of problem B and vice versa; and

(4) The technological coefficients in a particular column of problem A occur in the same row of problem B and vice versa (e.g. the coefficients in the second column of problem A and second row of problem B are both 1, 1, 2 respectively).

Given any maximisation/minimisation linear programming problem with all 'less than or equals'/'greater than or equals' constraints, it would be possible to write out the 'opposite' problem such that the foregoing relationships hold. By definition, one problem is said to be the dual of the other. It is in fact possible to construct a dual problem for a linear programming problem with a mixture of constraint types (including equalities) but this slightly

more difficult task is not dealt with here.

In order to investigate further the relationship between a linear programming problem and its dual, consider the final stage tableaus for problems A (repeated from Section II(iii)) and B:

PROBLEM A

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION
X_0	(1)	0	0	0	1	1	13
S_1	0	0	0	(1)	-3	1	1
X_1	0	(1)	0	0	2	-1	2
X_2	0	0	(1)	0	-1	1	3

PROBLEM B (Check this)

BASIS	Y_0	Y_1	Y_2	Y_3	S_1'	S_2'	R_1	R_2	SOLUTION
Y_0	(1)	-1	0	0	-2	-3	2-M	3-M	13
Y_2	0	3	(1)	0	-2	1	2	-1	1
Y_3	0	-1	0	(1)	1	-1	-1	1	1

It may be noted that:

(1) The optimal value of the objective in both problems is the same ($X_0 = Y_0 = 13$);

(2) The optimal values for the decision variables in Problem A ($X_1 = 2$; $X_2 = 3$) appear as the shadow cost values (i.e. in the R_1 and R_2 columns of the row labelled Y_0) in Problem B (recall M is set to zero); likewise, the optimal values for the decision variables in Problem B ($Y_1 = 0$; $Y_2 = 1$; $Y_3 = 1$) appear as the shadow cost values (i.e. in the S_1, S_2, S_3 columns of the row labelled X_0) in Problem A. This explains why these values are sometimes called the dual variable values;

(3) The optimal values for the slack variables in Problem A ($S_1 = 1$; $S_2 = 0$; $S_3 = 0$) appear (with negative signs because Problem B involves minimisation) beneath the variables Y_1, Y_2, Y_3 in the Y_0 row of the Problem B tableau; likewise, the optimal values for the slack variables in Problem B ($S_1' = 0$; $S_2' = 0$) appear (with positive signs because Problem A involves maximisation) beneath the variables X_1, X_2 in the X_0 row of the Problem A tableau.

The major implication of the foregoing interrelationships is that all the information gained from the final stage tableau for Problem A can be gained equally easily from the final stage tableau for Problem B. Thus, any linear programming problem may be solved indirectly by applying the simplex method to its dual.

Recall that a linear programming problem with many constraints is likely to require a greater solution time especially if these constraints are of the 'greater than or equals' type. The dual of such a problem would possess many decision variables but fewer constraints which would be of the 'less than or equals' type and would thus be more rapidly solved. This ability of the dual to provide an alternative problem, which may be solved more rapidly to yield the same information, represents one of its major uses.

IV SPECIAL CASES

(i) Introduction

Each of the linear programming problems considered previously has comprised a definable solution space and an optimal solution which occurs at a single outer point of that space. Algebraically, given n decision variables and m constraints (i.e. m slack variables), the optimal solution has always comprised exactly m constraint variables being greater than zero (in the final basis) with the rest equalling zero. Four instances in which one of the preceding conditions does not hold are now considered and their implications for the simplex method and its interpretation noted.

(ii) Constraint contradiction

Consider :

Maximise : $2X_1 + 3X_2$

s.t. : $2X_1 + 3X_2 \geq 18$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$

Graphical representation of the constraints which appear in the X_1, X_2 plane of Figure 6 indicates that these contain a contradiction : the first constraint requires that the optimal solution lies above AA' while the second and third constraints together require that it lies below BB' and CC' respectively. Thus there is no solution space or optimal solution.

In order to investigate how this condition could be recognised via the simplex method, consider the graphical representation of the extended problem i.e. with an additional variable, R_1 (Figure 6). The first constraint now requires that the optimal solution lies above the sloping plane AA'AA"; the second and third constraints together require that it lies below the planes formed by projecting BB' and CC' into the R_1 dimension. Overall, the solution space is bounded by YY'Y" as R_1 tends towards (but before it reaches) zero. Thus in the simplex method, R_1 must occur in the final basis i.e. be positive at optimality.

Any linear programming problem with contradictory constraints will by definition contain '≥' conditions which geometrically lie above '≤' conditions. Thus there will always be an extended problem solution space in which $R_1 > 0$ throughout. Thus, in the simplex method, $R_1 > 0$ (or, if there is a number of R variables, any $R > 0$) at the final iteration indicates a constraint contradiction.

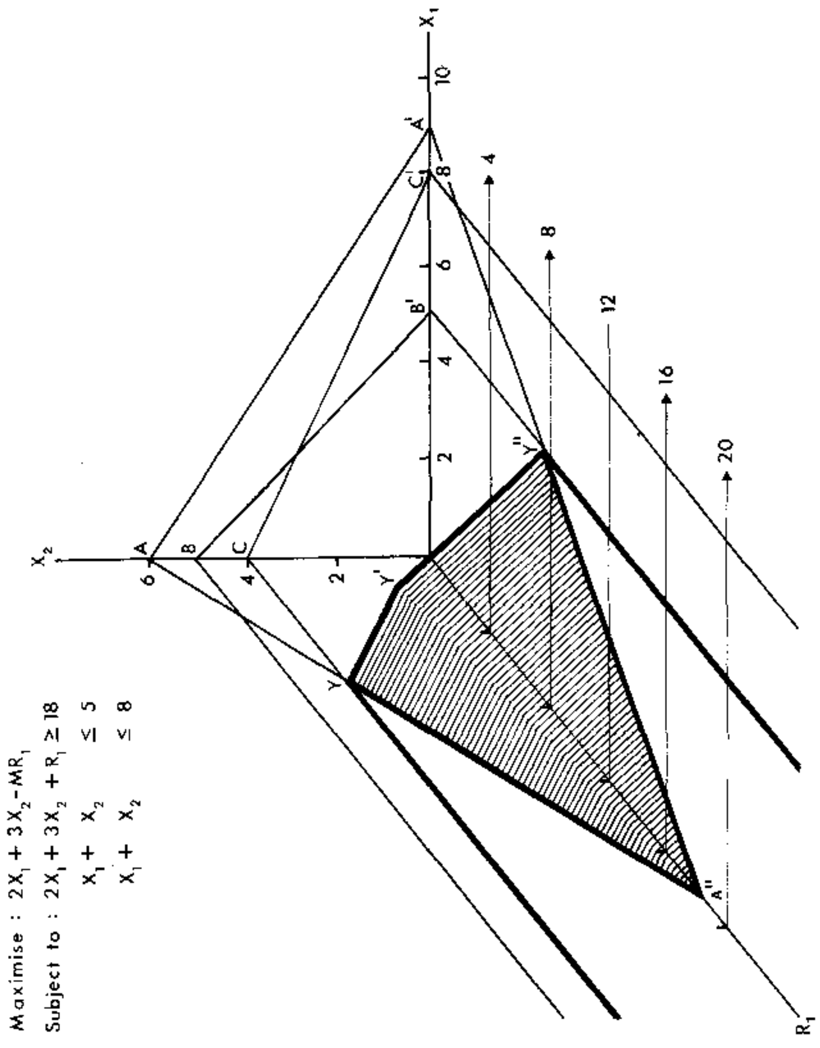

Fig. 6 Contradictory Constraints

(iii) Infinite Solution Space

Consider :

Maximise : $2X_1 - 3X_2$

s.t. : $X_1 - X_2 \leq 2$

$X_1 \leq 6$

which as illustrated (Figure 7) possesses a solution space of infinite size. A plot of the objective line (P_1P_1') indicates that this is maximised at A. Consider now the objective :

Maximise : $2X_1 + 3X_2$

A plot of the revised objective line (P_2P_2') indicates that this increases with distance away from the origin in the direction X_2; because the solution space is infinite in this direction, so too is the optimal solution.

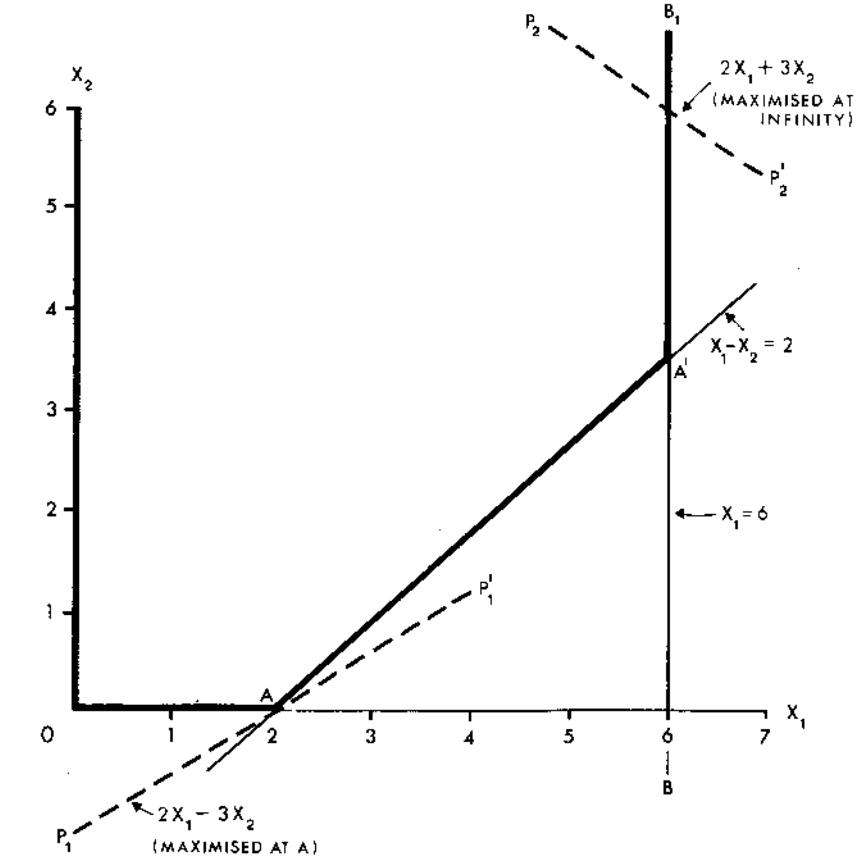

Fig. 7 Infinite Solution Space

From the point of view of the simplex method, what is required is, first, a means by which the existence of an infinite solution space can be recognised and, second, a method by which to check whether the optimal solution is infinite also. Consider the initial simplex tableau for the preceding problem :

BASIS	X_0	X_1	X_2	S_1	S_2	SOLUTION
X_0	(1)	-2	+3	0	0	0
S_1	0	1	-1	(1)	0	2
S_2	0	1	0	0	(1)	6

Note that the constraint coefficients in the X_2 column are all less than or equal to zero. Mathematically, it can be shown that if at any iteration in the simplex method (it happens to be the initial iteration in this instance), all the constraint coefficients in the column of a decision variable (X_2 in this case) are less than or equal to zero, then that decision variable may increase to infinity. In order to check whether $X_2 = \infty$ is in fact the optimal solution in this case, substitute $X_2 = \infty$ in the original objective. This yields $-\infty$ which, given that this is a maximisation problem, is obviously not the optimal solution. Thus the simplex calculations should continue. With $2X_1 + 3X_2$ as the objective, however, substitution of $X_2 = \infty$ yields an objective value of $+\infty$ which obviously is the optimal solution. Thus, in this case, the simplex calculations should cease.

(iv) Alternative Optimal Solutions

Consider :

Maximise : $2X_1 + 4X_2$

s.t. : $2X_1 + X_2 \leq 8$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$

The solution space for this problem is as for the farmer-crop problem (Figure 1(b)). The objective line would be parallel to the third (water) constraint (check this) and, when moved as far as possible from the origin in order to maximise profit, would coincide with the line segment CD. Thus there is a range of X_1, X_2 values which maximise the objective (i.e. there is a range of optimal solutions).

In order to investigate how the existence of alternative optimal solutions may be identified and dealt with <u>via</u> the simplex method, consider the initial tableau :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	-2	-4	0	0	0	0	-
S_1	0	2	1	(1)	0	0	8	8
S_2	0	1	1	0	(1)	0	5	5
S_3	0	1	[2]	0	0	(1)	8	4

which after the first iteration with X entering the basis and S_3-leaving yields :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	0	0	0	0	2	16	
S_1	0	3/2	0	(1)	0	-1/2	4	8/3
S_2	0	1/2	0	0	(1)	-1/2	1	2
X_2	0	1/2	(1)	0	0	1/2	4	8

which corresponds to C in Figure 1(b). The absence of negative values in the X_0 row demonstrates that this is an optimal solution; however, the existence of a zero coefficient for a variable outside the basis, X_1, indicates that if this variable is permitted to enter the basis, the value of the objective will not decrease and that an alternative, equally optimal solution exists with this variable in the basis. Permitting X_1 to enter the basis and pivoting yields :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	(1)	0	0	0	0	2	16	
S_1	0	0	0	(1)	-3	1	1	Negative
X_1	0	(1)	0	0	2	-1	2	1
X_2	0	0	(1)	0	-1	1	3	Negative

which corresponds to D in Figure 1(b). Note now that S_2 by virtue of a zero coefficient in the X_0 row may enter the basis; X_1 would leave to yield, once more, the penultimate simplex tableau. In practice, the interchangeability of the two solutions indicates that both are equally optimal; so too are all those on the line segment between them, i.e. on the line CD in Figure 1(b).

(v) Degeneracy

Mathematically, the term degeneracy applies loosely to any coincidence, chance occurrence or unique situation. In the context of linear programming, the most common degeneracy situation is where more than two constraints/axes intersect at an outer point of the solution space.

Consider :

Maximise : $2X_1 + 3X_2$

s.t. : $2X_1 + 5X_2 \leq 20$

$X_1 + X_2 \leq 5$

$X_1 + 2X_2 \leq 8$

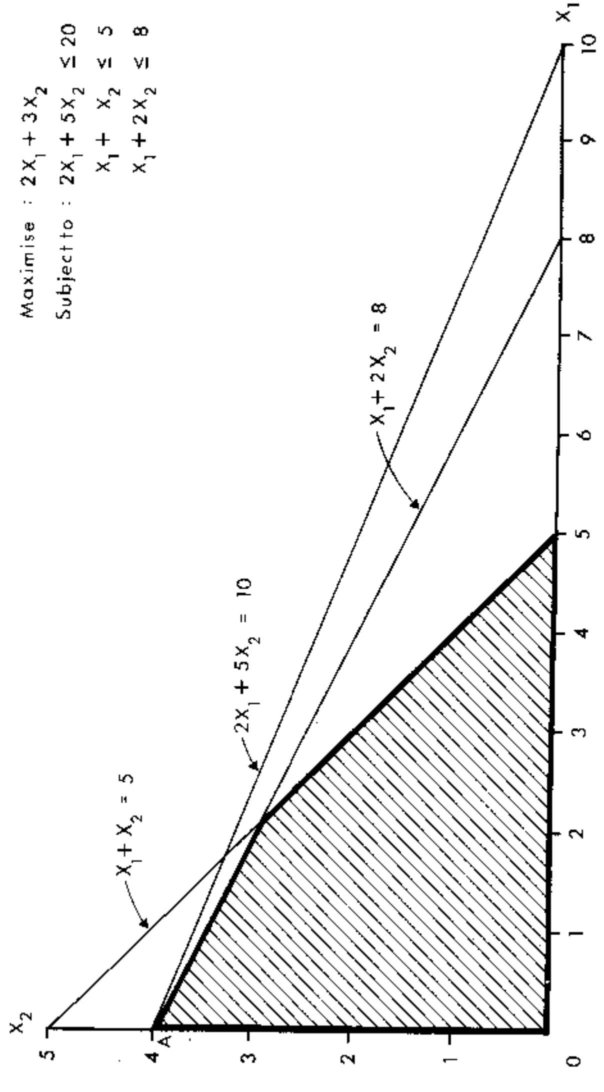

Fig. 8 Degeneracy

Graphical representation of the solution space (Figure 8) demonstrates that, by chance, the X_2 axis ($X_2 = 0$) and the first ($S_1 = 0$) and third ($S_3 = 0$) constraints all meet at A. The initial simplex tableau is :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION	RATIO
X_0	1	-2	-3	0	0	0	0	
S_1	0	2	5	1	0	0	20	4
S_2	0	1	1	0	1	0	5	5
S_3	0	1	2	0	0	1	8	4

On the first iteration X_2 enters the basis whereas, because of a tie between ratios, either S_1 or S_3 may leave. This indicates (as confirmed graphically) that as X_2 increases it is ultimately constrained simultaneously by the first and third constraints. Arbitrarily choosing S_1 to leave the basis gives :

BASIS	X_0	X_1	X_2	S_1	S_2	S_3	SOLUTION
X_0	1	-4/5	0	3/5	0	0	12
X_2	0	2/5	1	1/5	0	0	4
S_2	0	3/5	0	-1/5	1	0	1
S_3	0	1/5	0	-2/5	0	1	0

which corresponds to A (Figure 8). Notice that in this tableau one of the basis variables, S_3, equals zero in addition to zeros for the non-basis variables. The greater number of zero variables than expected signals degeneracy. Mathematically, it poses no particular problem and the simplex calculations may continue normally. (Check that the optimal solution is encountered after two further iterations.)

Two features of degeneracy vis a vis the simplex method deserve mention. First, with degeneracy the number of iterations required to reach the optimal solution depends to some extent on the arbitrary decisions taken concerning equally suitable variables to leave the basis. (Check that if S3 rather than S_1 leaves the basis in the initial iteration of the preceding example, the optimal solution is reached one iteration earlier.) Second, in exceptional circumstances (which are not discussed further here) it is possible for degeneracy to cause simplex calculations to enter an eternal loop throwing up alternately simplex tableaus which relate to the same degenerate solution point. The optimum is thus never reached. Such an event is extremely rare and it is not considered further here.

V EXAMPLES

(i) Introduction

The preceding sections together demonstrate how any linear programming problem may be solved and the results interpreted. In the context of an actual study, this process of solution and interpretation can take place only after an initial important task has been performed: that of identifying the problem concerned as a linear programming problem in the first place. As mentioned at the outset, the range of applications of linear programming problems is very great. To a large extent, the skill of recognising and expressing any one problem in terms of linear programming is one achieved through experience and through familiarity with previous studies. Three such studies are now discussed.

(ii) Human Diet

Gould and Sparks (1969) investigated the problem of finding for each of twenty-four town locations in Southern Guatemala the amounts of each of a set of available foods which should be included in a diet to minimise diet cost while meeting certain minimum nutritional requirements for calories, proteins and vitamins. Consider the case of three foods (eggs, oranges and tortillas) available at a particular location at costs of 9, 3 and 1 cent(s) per 100 grammes respectively. Each 100 grammes of a particular food purchased yields a known quantity of calories, protein and vitamin C (Table 4): the minimum requirements in a (daily) diet are 2700 calories, 65 grammes and 2000 units respectively. Following the logic employed in the farmer-crop problem, with X_1, X_2, X_3 equalling the amount of each food to be purchased, the minimum cost diet problem is :

$$\begin{aligned} \text{Minimise : } & 9X_1 + 3X_2 + X_3 \\ \text{s.t. : } & 160.0X_1 + 69.0X_2 + 201.0X_3 \geq 2700 \\ & 11.3X_1 + 0.8X_2 + 5.5X_3 \geq 65 \\ & 1.0X_1 + 200.0X_2 + 0.0X_3 \geq 2000 \end{aligned}$$

which is a linear programming problem.

Table 4. Human Diet Problem : Input Data

	Yield Per 100 Grammes		
	Eggs	Oranges	Tortillas
Calories	160.0	69.0	201.0
Protein	11.3	0.8	5.5
Vitamin C	1.0	200.0	0.0

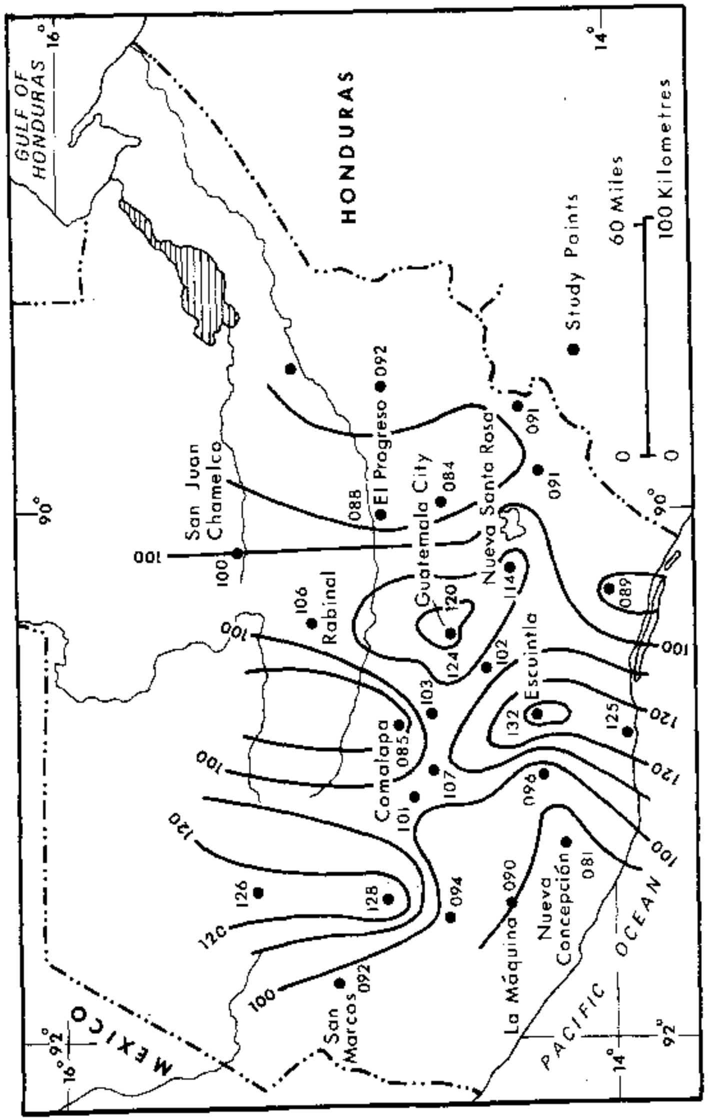

Fig. 9 Variation in the cost of a minimum cost diet in Guatemala (After Gould and Sparks)
Redrawn with permission from the Geographical Review, Vol. 59, 1969, 68.

Gould and Sparks' initial formulation actually involves forty foods (i.e. decision variables) and nine minimum dietary requirements (i.e. constraints). The cost of each of the forty foods (and thus the form of the objective) differs for each of the twenty-four town locations and the linear programming problem is solved for each in turn. The regional variation in the cost of the least cost diet is presented in the form of an isoline map (Figure 9) and demonstrates, for example, the intensity of the price peaks at the large urban agglomerations caused by increased demand and thus higher food prices. Gould and Sparks subject their initial solutions to two types of sensitivity analysis. First, additional constraints which reflect both further dietary requirements and preferences based on cultural differences between towns are added; the problem is re-solved for the twenty-four towns and the isoline map re-plotted. Secondly, the amounts by which the prices of foods not in the optimal diet would have to fall in order to be so included is investigated. These two types of sensitivity analysis correspond respectively to the fifth and second types listed in Section III(i).

Gould and Sparks' study demonstrates both the flexibility offered by linear programming for investigating geographical problems of practical significance and some of the difficulties that can arise when employing the method. Concerning the latter, it can be noted that the input data requirements for a linear programming study are often fairly severe, comprising in the Guatemala case the price per 100 grammes of forty foods at twenty-four locations, the number of calories, vitamins, etc. yielded by 100 grammes of each of these foods, and minimum dietary requirements. Fortunately, all of this information was available to Gould and Sparks; sometimes, this is not the case. Another difficulty which often arises with linear programming concerns the sheer volume of data involved. Gould and Sparks' initial formulation involves forty decision variables and nine constraints which would not be considered to be particularly large. Yet, to solve the resultant programming problem by hand using the methods discussed previously would prove exceedingly time consuming. For this reason, many real-world linear programming problems are solved by computer (Section VI(i)).

(iii) Irrigation

Soltani-Mohammadi (1972) considers the problem of determining which of two types of irrigation technique (conventional surface and sprinkler) should be employed in the Ghazvin area west of Teheran, Iran and, given the technique chosen, the amounts of each of twelve crops that should be grown in order to maximise profit. The problem is approached by solving two linear programming problems, one for each irrigation technique, and comparing the results.

If c_j is the profit per unit and X_j the amount of crop j grown, then, mathematically, the objective is :

$$\text{Maximise : } c_1 X_1 + c_2 X_2 + \ldots + c_{14} X_{14}.$$

Consider the case of conventional surface irrigation. To grow crops using this method involves using known quantities of five resources: land, water, skilled labour, unskilled labour and machinery. Given the amount of resource i used up per unit of crop j produced (a_{ij}) and the total amount of this resource available (b_i), then, following the logic of the farmer-crop problem, five constraints of the form :

$$a_{i1} X_1 + a_{i2} X_2 + \ldots + a_{i14} X_{14} \leq b_i \; ; \; i = 1, \ldots, 5$$

may be defined. Soltani-Mohammadi introduces nine additional constraints relating to various market and crop rotation limitations (e.g. concerning the maximum amounts of certain crops that may be grown). Solution yields the amount of each crop that should be grown to yield maximum profit under conventional irrigation. Adjusting the a_{ij} values in the constraints to reflect sprinkler technology yields a revised solution and maximum profit level. In the case of Soltani-Mohammadi's example, the results demonstrate that sprinkler technology would return the higher profit. The optimal sprinkler technology solution is constrained by the water and skilled labour availability constraints (indicated by slack variable values of zero associated with these) and the extent to which profit would increase with a unit increase in these resources is investigated via the methods of Section III(ii). Soltani-Mohammadi also calculates the extent to which the availability of water and skilled labour would have to alter in order to cause conventional irrigation to return the higher profit.

(iv) Urban Development

An area in which linear programming has been employed to advantage by a number of workers is that of determining the amount of various landuses that should be permitted in a specific urban development scheme in order to fulfil some objective such as to maximise total return from sales while keeping within the bounds of stated planning and other criteria. Laidlaw (1972) (for further examples see Herbert and Stevens (1960) and Heroux and Wallace (1975)) discusses a number of examples commencing with the following (slightly altered) hypothetical case which demonstrates the general approach.

Ten acres of a town are to be developed with a mix of middle income and high income housing. The former sells for £15 000 per house with 20 houses per acre (0.05 acres per house) and the latter for £20 000 per house with 12 houses per acre (0.083 acres per house). If X_1, X_2 are the total number of middle and high income houses to be constructed and it is required that return from sales be maximised, then the objective is :

$$\text{Maximise}: \quad 15000\,X_1 + 20000\,X_2$$

Upper and/or lower limits on the number of a specific housetype or the total number of houses ($X_1 + X_2$) that may be built can be expressed by such constraints as :

$$X_1 \geq 70$$
$$X_1 + X_2 \leq 155$$

A constraint requiring that (say) the number of middle income houses is not more than twice the number of high income houses is given by :

$$X_1 \leq 2X_2$$
$$\text{i.e.} \quad X_1 - 2X_2 \leq 0$$

Another constraint must be that the total amount of land developed does not exceed that available i.e.

$$0.05X_1 + 0.083\,X_2 \leq 10$$

A further constraint could relate to the types of resident in the final development. If it is known that 22 per cent and 16 per cent of the middle and high income households respectively will be occupied by single persons and if it is required that (say) no more than 20 per cent of households in

the final project be occupied by single persons then :

$$0.22X_1 + 0.16\,X_2 \leq 0.2\,(X_1 + X_2)$$
$$\text{i.e.} \quad 0.02X_1 - 0.04\,X_2 \leq 0$$

A further constraint could relate to mortgage payments; if 97 per cent of the cost of a low income house (£14 500) and 90 per cent of the cost of a middle income house (£18 000) is available through a mortgage scheme and if £2 475 000 have been set aside as the maximum amount available for mortgage purposes, then :

$$14500\,X_1 + 18000\,X_2 \leq 2475000.$$

The preceding problem is a linear programming problem and indeed, by virtue of having two decision variables, could be solved graphically. More generally and realistically, there could be any number of landuse types (decision variables) and constraints. Laidlaw discusses a number of real world applications including development projects in Jersey City, New York and Baltimore and, in particular, highlights the practical significance of the special cases discussed in Section IV. One most valuable contribution of linear programming in a situation where there are many decision variables and constraints can be to indicate whether, in fact, there is any feasible solution. Again, linear programming can indicate whether, by virtue of the existence of alternative optimal solutions, the planner possesses a degree of latitude in fulfilling his objective. Sometimes in a planning context, different interest groups claim that different objectives and/or constraints are most appropriate; it can emerge via linear programming that both objective/constraint sets lead to highly similar solutions and thus that to argue between them is unnecessary.

VI FURTHER TOPICS

(i) Computer Programs

Most real-world linear programming problems involve too many decision variables and constraints for solution by hand. Fortunately, as demonstrated, applying the simplex method comprises performing a set of precisely defined mathematical calculations repetitively without the need for human judgement. Thus it is amenable to programming and solution by computer.

Most computer installations possess linear programming packages with information as to how these may be used: such packages usually solve a given problem and then, if required, perform many types of sensitivity analysis. Before employing the computer to deal with a specific linear programming problem, the precise information required should be defined carefully; the temptation of obtaining as much data output as possible irrespective of its meaning or relevance should be avoided.

(ii) Notation

Consider a linear programming maximisation problem, with n decision variables and m constraints of the '$\leq$' type. Using the notation employed in Section V(iii), the problem is:

$$\begin{aligned} \text{Maximise : } & c_1 X_1 + c_2 X_2 + \ldots + c_n X_n \\ \text{s.t. : } & a_{11} X_1 + a_{12} X_2 + \ldots + a_{1n} X_n \leq b_1 \\ & a_{21} X_1 + a_{22} X_2 + \ldots + a_{2n} X_n \leq b_2 \\ & \vdots \\ & a_{m1} X_1 + a_{m2} X_2 + \ldots + a_{mn} X_n \leq b_m \\ & X_1, X_2, \ldots, X_n \geq 0 \end{aligned}$$

Mathematically, this may be summarised as :

$$\text{Maximise : } \sum_{j=1}^{n} c_j X_j$$

$$\text{s.t. : } \sum_{j=1}^{n} a_{ij} X_j \leq b_i \; ; i=1, 2, \ldots, m$$

$$X_j \geq 0 \; ; j=1, 2, \ldots, n.$$

This is the usual format in which linear programming problems are presented in the literature although the actual letter notation varies. Following Section III(iii), the dual of the preceding problem involves m decision variables and n constraints. Denoting the former as $Y_1, Y_2, \ldots, Y_m$, the dual is:

$$\text{Minimise : } \sum_{i=1}^{m} b_i Y_i$$

$$\text{s.t. : } \sum_{i=1}^{m} a_{ij} Y_i \geq c_j \; ; j=1, 2, \ldots, n$$

$$Y_i \geq 0 \; ; i=1, 2, \ldots, m$$

This is the usual format in which the dual is presented in the literature.

(iii) The Transportation and Related Problem's

A previous publication in the CATMOG series (Hay, 1977) considers what is known as the transportation problem of linear programming. The objective comprises finding the pattern of flows for a particular product between a set of supply points (e.g. factories) and demand points (e.g. towns) such that total transportation costs are minimised. Consider the case of two supply points, A_1 and A_2, with a_1 and a_2 units of product available for shipment and of two demand points, N_1 and N_2, requiring n_1 and n_2 units of product respectively. Then, following the notation and approach of Hay (pp. 4-7), if m_{ij} is the cost of transporting one unit of produce from A_i to N_j, f_{ij} is the number of units to be sent from i to j and the total supply available, $(a_1 + a_2)$ equals the total demand requirement, $(n_1 + n_2)$, the problem is:

$$\begin{aligned} \text{Minimise : } & m_{11} f_{11} + m_{12} f_{12} + m_{21} f_{21} + m_{22} f_{22} \\ \text{s.t. : } & f_{11} + f_{12} = a_1 \\ & f_{21} + f_{22} = a_2 \\ & f_{11} + f_{21} = n_1 \\ & f_{12} + f_{22} = n_2 \end{aligned}$$

where $f_{11}, f_{12}, f_{21}, f_{22}$ (≥ 0) are the decision variables. Note that this is a linear programming problem as defined in Section I(iii) and that it could therefore be solved using the simplex method. Note too, however, that the problem possesses two special features (which would hold true for any number of supply and demand points provided total supply equals total demand): first, the coefficients of the decision variables in the constraints all equal either one or zero and, second, all the constraints are equalities. In a mathematical sense, the transportation problem thus represents a special (and simpler) case of linear programming and it is for this reason that the more straightforward methods described by Hay can be employed to solve it.

It should be noted that a number of other linear programming problems exist which, by virtue of a special structure can, if required, be solved using more straightforward methods. A number of these which are of relevance to geographers are discussed by Scott (1971).

APPENDIX

SOLVING SIMULTANEOUS LINEAR EQUATIONS

Consider the pair of equations :

$$\left.\begin{aligned} 2X_1 + 3X_2 &= 8 \\ 3X_1 + 5X_2 &= 13 \end{aligned}\right\} \text{------------ (A1)}$$

Suppose that the numerical values of X_1 and X_2 which cause both equations to hold true are required. The usual method of finding these values is to begin by eliminating one of the variables using multiplication and subtraction. Thus, multiplying the first equation in (A1) across by three and the second across by two yields :

$$\left.\begin{aligned} 6X_1 + 9X_2 &= 24 \\ 6X_1 + 10X_2 &= 26 \end{aligned}\right\} \text{------------ (A2)}$$

Subtraction eliminates X_1 and gives :

$$-X_2 = -2$$

or $X_2 = 2$. X_1 may now be found by substituting $X_2 = 2$ in either of the original equations (A1) which gives :

$$X_1 = 1$$

That $X_1 = 1$, $X_2 = 2$ are indeed the numerical values of X_1 and X_2 which cause both of the original equations to hold true can be checked by substituting these values in the left hand sides of (A1) which yields :

$$2(1) + 3(2) \overset{\checkmark}{=} 8$$
$$3(1) + 5(2) \overset{\checkmark}{=} 13$$

Finding the numerical values of variables such as X_1 and X_2 which together cause a set of equations such as those in (A1) to hold true is often referred to as solving <u>simultaneous linear equations.</u> In performing this task above, use was made of an important mathematical property namely : <u>equations can be multiplied or divided across by chosen constants without</u>

altering their solution. Note that when $X_1 = 1$, $X_2 = 2$ are substituted in the left hand sides of (A2), these equations also hold, which demonstrates the property. The equation sets (A1) and (A2) are said to be equally valid.

Another important mathematical property of simultaneous linear equations is that : an equation can be multiplied or divided across by a constant and then added to or subtracted from another equation without altering the solution of the resulting equations. Thus in (A1), subtracting (say) twice the first equation from the second yields the revised equation set :

$$\left.\begin{aligned} 2X_1 + 3X_2 &= 8 \\ -X_1 - X_2 &= -3 \end{aligned}\right\} \text{------------ (A3)}$$

Substituting $X_1 = 1$, $X_2 = 2$ shows that (A1) and (A3) are equally valid thus demonstrating the property.

Consider now the task of rewriting the equation set (A1) such that the left hand sides are :

$$\left.\begin{aligned} 1X_1 + 0X_2 &= ? \\ 0X_1 + 1X_2 &= ? \end{aligned}\right\} \text{------------ (A4)}$$

Placing the equations in this form can be achieved by judicious use of the two rules stated previously. First, to create a coefficient of '1' for X_1 in the first equation, divide it across by '2' :

$$\left.\begin{aligned} 1X_1 + 3/2\ X_2 &= 4 \\ 3X_1 + 5\ X_2 &= 13 \end{aligned}\right\} \text{------------ (A5)}$$

Now, to create a coefficient of '0' for X_1 in the second equation, multiply the first equation in (A5) across by three and subtract it from the second :

$$\left.\begin{aligned} 1X_1 + 3/2\ X_2 &= 4 \\ 0X_1 + 1/2\ X_2 &= 1 \end{aligned}\right\} \text{------------ (A6)}$$

Now, to create a coefficient of '1' for X_2 in the second equation, multiply this equation across by 2 :

$$\left.\begin{aligned} 1X_1 + 3/2\ X_2 &= 4 \\ 0X_1 + 1\ X_2 &= 2 \end{aligned}\right\} \text{------------ (A7)}$$

Finally, to create a coefficient of '0' for X_2 in the first equation, multiply the second equation across by '3/2' and subtract it from the first :

$$\left.\begin{aligned} 1\ X_1 + 0\ X_2 &= 1 \\ 0\ X_1 + 1\ X_2 &= 2 \end{aligned}\right\} \text{------------ (A8)}$$

The left hand sides of (A8) are now in the format required by (A4). Furthermore, this format allows the solution values for X_1 and X_2 to be read off directly for, ignoring the zero coefficients, the equations (A8) state : $X_1 = 1$, $X_2 = 2$. For this reason, the task of solving simultaneous linear equations may, if required, be viewed as one of using the two rules stated above to adjust the left hand sides of the equations to a format which enables the solutions to be read off directly.

For two equations and two unknown variables, use of the approach of (A4)-(A8) to determine the solution values may seem complicated compared with that of (A1)-(A2). Its advantage becomes evident when there are more than two equations and unknowns. Consider :

$$\left.\begin{aligned} 2X_1 + 3X_2 + X_3 - X_4 &= 5 \\ 3X_1 + 5X_2 + 2X_3 - 3X_4 &= 3 \\ X_1 + X_2 + X_3 + X_4 &= 8 \\ X_1 + 3X_2 - 2X_3 + X_4 &= 9 \end{aligned}\right\} \text{------------ (A9)}$$

Here, it is difficult to envisage how the values of X_1, X_2, X_3, X_4 which enable all four equations to hold simultaneously could be determined using the approach discussed at the outset of the appendix. Thus it is more convenient to view the problem as one of adjusting the formats of the left hand sides of the equations to give ultimately :

$$\left.\begin{aligned} 1X_1 + 0X_2 + 0X_3 + 0X_4 &= ? \\ 0X_1 + 1X_2 + 0X_3 + 0X_4 &= ? \\ 0X_1 + 0X_2 + 1X_3 + 0X_4 &= ? \\ 0X_1 + 0X_2 + 0X_3 + 1X_4 &= ? \end{aligned}\right\} \text{------------ (A10)}$$

Again, this may be achieved by judicious application of the two mathematical rules stated previously. First, to create a coefficient of '1' for X_1 in the first equation, divide this across by '2'. Next, to create zero coefficients for X_1 in the other rows, multiply the revised first row by 3, 1 and 1 and subtract it from the second, third and fourth rows respectively. Now repeat the process in turn on X_2, X_3 X_4. (The final answer is $X_1 = 1$; $X_2 = 2$; $X_3 = 1$; $X_4 = 4$).

A final important point to note concerning simultaneous linear equations is that in order to obtain numerical values for the unknowns, the number of equations must at least equal the number of unknowns. Thus in the case of :

$$\left.\begin{aligned} 2X_1 + 3X_2 + X_3 - X_4 &= 5 \\ 3X_1 + 5X_2 + 2X_3 - 3X_4 &= 3 \end{aligned}\right\} \text{------------ (A11)}$$

unique values for X_1, X_2, X_3, X_4 cannot be determined for there are four unknowns but only two equations. The determination of values for the unknowns in (A11) only becomes possible if some of these values are given: e.g. if it is given that $X_3 = X_4 = 0$, then, automatically (because there are now effectively two equations and two unknowns) $X_1 = 16$ and $X_2 = -9$; again, if it is given that in (A11) $X_2 = X_4 = 0$ then, automatically, $X_1 = 7$ and $X_3 = -9$.

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